Discontinuous Galerkin Method on Tetrahedral Elements for Aeroacoustics

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# DISCONTINUOUS GALERKIN METHOD ON TETRAHEDRAL ELEMENTS FOR AEROACOUSTICS 

## PROEFSCHRIFT

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## GENERAL INTRODUCTION

### 1.1 Sound and acoustics

Historically both the words sound and acoustic are related to the sensation of hearing (Goth [35]). The word sound is derived from the Latin word sonus which means noise. The word acoustic comes from the Greek word akousitkos which means pertaining to hearing. The oldest defenition of sound is therefore the physiological definition of sound: sound is the sensation produced in hearing organs by vibrations. Any unwanted sound is called noise. Often these unwanted sounds are erratic, intermittent and random in nature.

In modern-day physics, sound is defined as a succession of compressions and rarefactions (vibrations) in a fluid that propagate away from the acoustic source region (Howe [41]). By this definition, the term sound does not only imply phenomena in air responsible for the sensation of hearing, but all phenomena that are governed by analogous physical principles (Howe [41]). The term sound not only applies to the medium air; one may speak of underwater sound, sound in solids etc. Sound is characterized by its intensity, expressed in decibel $(d B)$, and its frequency, expressed in Hertz $(H z)$. On the logarithmic decibel scale, an increase of 3 dB means that the intensity of noise has doubled. The human ear is able to distinguish a large range of noise intensity, from $1 d B$ to $120-125 d B$. The latter is the start of the threshold of pain.

Acoustics is defined as the science of sound, including its production, transmission and effects (Pierce [63]). Sound may be produced by vibrating mechanical structures, agitated regions of turbulent flow, the mixing of flows of different temperature, and so forth. There are a great number of subjects covered by acoustics, as is illustrated by Fig.(1.1) ${ }^{1}$. As illustrated in Fig.(1.1), the subdivision in acoustical topics is usually based on the medium in which sound is propagating, or the impact sound has on life or its surroundings, or the source of sound responsible for the generation, etc. (Pierce [63]).

Two topics, part of the scope of acoustics, which we will encounter later in this thesis are vibrational acoustics and aeroacoustics. In vibrational acoustics, sound is generated by a vibrating structure. When sound is produced as a byproduct of an airflow, we are dealing with aeroacoustics. In the field of aeroacoustics the interaction is studied between acoustic waves and the flow itself. The foundations of the field of aeroacoustics were developed by

[^0]

Figure 1.1: Scope of acoustics. Figure taken from Pierce [63].

Lighthill ([54], [55]) in the 1950's. Lighthill who also introduced the term aerodynamic sound. Lighthill identified aerodynamic sources of sound, by representing the complex fluid mechanical process that acts as an acoustic source by an acoustically equivalent source term for the non-homogeneous wave equation.

Based on the auditory capabilities of humans, sound can also be grouped, according to the frequencies $(f)$ involved, into the audible and the inaudible range. Each of these frequency ranges has its own specialized research topics:

- Audible sound: The auditory range of humans consists of perception between frequencies of 20 Hz (the deepest perceivable bass note) and 20 kHz (the highest perceivable tone). Although audible sound can be enjoyed as music (wanted sound), sound produced in the audible range has received a lot of attention lately because of another reason, i.e. noise (unwanted sound). Often these unwanted sounds are erratic, intermittent and random in nature. Noise produced in the audible range can interfere with speech, lead to hearing impairment (when the intensity or loudness of the sound is high enough), furthermore it can be a major source of stress, with physiological reactions including accelerated heart rhythm, increased amounts of hormones produced and dialation of the pupils ([87]). The policy of the European Union and that of the national governments is aiming at reducing noise pollution, primarily in the audible
range, which has lead to an enormous increase in the interest in acoustics.
Other research topics in the audible range are related to, for example, communication, speech, hearing, sound produced by musical instruments as well as room and theater acoustics.
- Infrasound: Frequencies too low to be heard by a normal person, i.e. $f \leq 20 \mathrm{~Hz}$. Most infrasonic research is based on frequencies of 10 Hz or below, to 0.001 Hz (used in monitoring earthquakes), although formally the lower limit of the infrasonic domain is not strictly defined. When considering extremely low frequencies, of the order of $g / c$, where $g$ is the acceleration due to gravity and $c$ is the speed of sound, the influence of gravity, which in general has a negligible influence, can no longer be neglected )Pierce [63]).

Infrasonic waves travel for long distances through air and particularly through earth. Some organisms, such as elephants, are capable of employing infrasound and seismic waves (energy waves) as means of communication, which may influence them to be sensitive to changes within the earth, under the surface, as earthquakes occur.

It is less well known that infrasound can be harmful to living organisms (Castelo Branco [18]). Research has shown that Low Frequency Noise (LFN), $f \leq 100 \mathrm{~Hz}$, including infrasound, can be an agent of disease (Castelo Branco [18]). Many apparatuses commonly used in working environments, such as fans, blowers, compressors, etc., produce noise with frequencies below 100 Hz . Often the associated sound pressure levels exceed $100 d B$. Sound produced at these frequencies and with such intensities hardly impairs the human auditory organs (our ears are not very sensitive at these frequencies), but it is, however, known to induce human body vibration (Castelo Branco [18], Takahashi et al. [74]). It is speculated that though the level of this vibration is not very high, long-term exposure might lead to health problems of an exposed worker (Takahashi et al. [74]). Health problems have been reported in workers who have been exposed to high-intensity low-frequency noise for more than ten years, such as military and civilian pilots and aircrews, aeronautical mechanics and technicians, and, more recently in a civilian population exposed to military training exercises (Castelo Branco [18]).

- Ultrasound: Frequencies too high to be heard by a normal person; $f \geq 20 \mathrm{kHz}$. Sonar (Sound Navigation and Ranging) or echolocation, which works in the $k H z$ range, is a well-known topic related to the ultrasound range. Another example is the treatment of kidney stones. Kidney stones are broken in small fragments when ultrasonic shock waves are focussed on the stones.


### 1.2 Computational Aeroacoustics

The study of the problems of aeroacoustics using computational techniques is called Computational Aeroacoustics (CAA). CAA is a rapidly growing branch in fluid mechanics, to a large extent because of the governmental policies on the reduction of sound emissions. this relates especially to the noise produced by aircraft, but also trucks, cars (highways) trains etc.

Much of the advance has also been driven by the availability of more powerful computational resources.

The goals of CAA are to enable aeroacoustic predictions in for variety of flows and engineering devices, and to advance our understanding of the physics of the noise generation and propagation (Colonius [25], Tam [77]). In CAA, like in Computational Fluid Dynamics (CFD), there is a great variety of physical systems, physical models, numerical algorithms and solution strategies in use.

Although CFD has become an indispensable and reliable method for solving aerodynamic problems, most CFD methods are not well-suited for solving aeroacoustic problems. This because the nature, characteristics and objectives in aeroacoustics are distinctly different from those in aerodynamics (Tam [77]). Aeroacoustic problems are by definition time-dependent, whereas aerodynamic problems are frequently described as time-independent, i.e. steady (Tam [76], [77]). Also, in most aeroacoustic problems there is large difference in the length scales in the source region (near field) and in the region of propagation (farfield). Furthermore, the frequencies are usually relatively high and the propagation distances are relatively large. Inherently to using numerical methods in CAA (like in CFD), numerical dissipation and dispersion are introduced. Numerical dissipation unphysically dampens the vibration amplitude of the sound waves and numerical dispersion alters the propagation velocity in a unphysical manner. The longer the propagation distances and the higher the frequencies involved the more apparent these unphysical influences become. Higher-order methods, higher than the usually first or second-order CFD methods, can be used to remedy these problems (Tam [76], [77]).

Aeroacoustic problems often involve large domains. The physical domain might even be given by infinite or semi-infinite domains. Not only must the computational domain be of finite dimensions, also with a larger computational domain the costs of a computation increases primarily because, in contrast to CFD, for CAA the far field has to have a high-resolution grid. It is therefore common use in CAA to truncate the computational domain at some finite distance from the source region. The boundary conditions which should be applied at the boundary of the truncated computational domain should be such that the solution of the modified problem is as close as possible to the solution of the original problem. The truncation should, if possible, be made in a region where the flow can be regarded as small disturbance to a uniform flow (Colonius [25]). Even when the boundary conditions have been treated with care the boundary conditions can give rise to unphysical reflections which contaminate the numerical solution. The development of proper, so-called, non-reflecting boundary conditions is in general very difficult and a topic of active research.

Fig.(1.2) shows a problem related to an open domain, indicating the near-field and farfield region. When both the acoustic source region (near field) and the region of acoustic propagation (far field) are unknown, the following computational strategies might be used (Colonius [25]):

- Direct computation: The compressible Navier-Stokes equations are solved on the entire domain. The unsteady near field (turbulence), i.e. the acoustic source region, is either resolved on the mesh (Direct Numerical Simulation (DNS)) or the small scales are modelled, e.g. Large Eddy Simulation (LES), Very Large Eddy Simulation (VLES) or unsteady Reynolds Averaged Navier-Stokes (RANS). The computational domain extends at least a few acoustic wave lengths.


Figure 1.2: Open domain with acoustic source region (near field) and the region of acoustic propagation (far field).

- Hybrid approach: Several lower-dimensional models (derived from the Navier-Stokes equations) are used either simultaneously in different parts of the domain, or subsequently in the whole domain.
- Matching: The acoustic source region or near field is obtained employing direct computation (DNS, LES, VLES, RANS), i.e. by solving the compressible NavierStokes equations directly or in an averaged form. The computational domain used for the direct computation is truncated somewhere in between the acoustic near and far field. For the direct computation artificial boundary conditions are imposed at this intermediate domain boundary. The solution at the boundary is matched to a set of simplified, or reduced, equations, e.g. the linearized Euler equations, for the remainder of the computational domain which extends at least several acoustic wave lengths into the far field.
- Kirchhoff surface: The compressible Navier-Stokes equations are solved on a computational domain in the acoustic near field and artificial boundary conditions are imposed at the domain boundary. The solution at the boundary of this near field domain is then passed to a Kirchhoff-surface method. In such a method, at this boundary an integral relation is used to relate the pressure and its derivatives along the surface to the pressure in the acoustic field outside the surface. In the integral formulation it is assumed that at the Kirchhoff surface the flow is linear. This implies that the boundary has to be located at a distance sufficiently far away from the source region. It is, however, difficult to estimate a priori where the disturbances become small enough so that the linearized equations can be applied.
- Acoustic analogy: The near field is obtained by direct numerical simulation. Along some boundary intermediate to the acoustic near and far field the domain is truncated and artificial boundary conditions are imposed. A suitable acoustic analogy, for example Lighthill's acoustic analogy, is used to relate the acoustic field at points exterior to the computational domain to volume integrals of data inside the domain.
- Hybrid methods: Hybrid methods couple nearly-incompressible unsteady flow simulations to an acoustic solver (Lele [52]). For example, Hardin and Pope [38],
propose a scheme of splitting the flow variables into (nearly-) incompressible and acoustic parts. Lele [52] writes about this approach: 'The nearly incompressible flow is at leading order strictly incompressible. The pressure variations (required to maintain a strictly divergence-free velocity field) in this incompressible flow are linked to an isentropic density perturbation. This nearly incompressible flow description is subtracted from the exact nonlinear compressible flow equations, and the resulting nonlinear set is viewed as a set appropriate for the acoustic field and discretized on an acoustic mesh, which is chosen with a suitably large mesh spacing so that only the expected large-scale acoustic field is represented. The nearly-incompressible flow is advanced on a different mesh".


### 1.3 Objective, motivation and approach

The objective of this thesis is to develop and verify a numerical method for the propagation of acoustic information through a three-dimensional domain which may involve complex geometries.

As known from literature (see for example Pierce [63]), the produced acoustic disturbances are often so small that it is justified to linearize the equations of motion about some mean flow solution, as a first approximation. Although the attenuation of sound waves increases with increasing frequency of the sound waves, the effect of viscosity on the propagation of sound is also assumed to be negligible, i.e., when propagation distances are not too large (for example in the order of kilometers). The propagation of sound can therefore be described, as a first approximation, by the linearized Euler equations (LEE).

For numerically solving the LEE, three methods which have been considered are discussed below, namely the Dispersion Relation Preserving (DRP) finite-difference scheme of Tam \& Webb ([75], [77]), the (Weighted) Essentially Non-Oscillatory (ENO/WENO) schemes, introduced by Harten, Osher and others ([2], [39]) and the Discontinuous Galerkin finite-element method, as developed by Atkins \& Shu ([5], [6]).

## - Dispersion Relation Preserving finite-difference scheme

The Dispersion Relation Preserving (DRP) finite-difference scheme of Tam \& Webb, is an optimized high-order finite difference scheme. The subject of the optimization is the dispersion relation, the functional relation between the angular frequency of the waves and the wave numbers of the spatial variable (Tam \& Webb [75]). DRP schemes are designed such that the dispersion relation of the finite-difference scheme approximates the dispersion relation of the original partial differential equations as closely as possible over a range of wave numbers which is as large as possible. First derivatives are approximated by central (or symmetric) finite-difference stencils. As explained by Tam \& Webb [75] such a symmetric stencil is preferred, because asymmetric stencils may allow numerical instabilities to occur. The DRP scheme has been successfully applied by many authors to many different acoustic problems. However, because of the size of the stencil, which depends on both the required formal accuracy and the desired accuracy of the dispersion and dissipation errors, the handling of boundary conditions is not straight-forward. Near boundaries either ghost cells have to be introduced or the stencil has to be allowed to be asymmetric. Most importantly, the method requires the
mesh to be regular. The DRP scheme is therefore not very well suited for application to problems involving complex geometries and was therefore not chosen.

## - Essentially Non-Oscillatory schemes

Essentially Non-Oscillatory (ENO) and Weighted Essentially Non-Oscillatory (WENO) are high-order-resolution schemes constructed for hyperbolic conservation laws. The basic feature of ENO schemes, given a mesh (which may be unstructured) and an algorithm, is to reconstruct a priori at any order of accuracy a given possibly unsmooth function either, from its node values or from control-volumes around its nodes (Abgrall [2]). ENO and WENO schemes provide uniformly high-order of accuracy up to discontinuities and are not reduced to first-order schemes in the vicinity of local extremes. Thus, they are suitable for numerical simulation of flows with a complex, nontrivial behavior of the solution in smooth regions. To calculate the fluxes at the cell boundaries, the 'smoothest" stencil is chosen from several possible stencils, where the value of the higher divided differences of fluxes is being used as a smoothness indicator. In WENO schemes all candidate stencils are taken into account. The weights with which the different possible stencils are included in the WENO scheme are chosen such that stencils in which the solution is smooth have a high weight and stencils in regions near discontinuities have a nearly zero weight. The treatment of boundary conditions is not straightforward. Near domain boundaries the choice of candidate stencils becomes limited and sometimes the reconstruction cannot be performed. Because the method is not compact, regularity of the mesh influences the accuracy. Because of these two points the method was not considered further here.

## - Discontinuous Galerkin finite element method

The Discontinuous Galerkin finite element method is an extremely compact method which is applicable to both structured and unstructured meshes. The solution is approximated by a local polynomial expansion in each element of the computational mesh and the solution is not required to be continuous over element interfaces. The size of the computational stencil is independent of the desired order of the method. DG methods of arbitrarily high formal accuracy can be obtained by suitably choosing the degree of the approximating polynomials (Cockburn et al. [19]). Because of the compactness of the scheme, it is highly parallelizable, it is very well suited for handling complex geometries and requires a simple treatment of the boundary conditions. In addition, the method is not sensitive to mesh irregularities. DG methods can furthermore easily be combined with local grid refinement (h-refinement) and local polynomial degree-refinement strategies (p-refinement). A drawback of the method is the increasing number of unknowns per element with increasing order of the method.

In the present thesis, the Discontinuous Galerkin finite-element method, as developed by Atkins \& Shu ([5], [6]), is used for the spatial discretization of the 3D LEE. For the time integration the multi-stage low-storage Runge-Kutta algorithm is employed. The LeE are solved on tetrahedral meshes employing the Discontinuous Galerkin method for the spatial discretization. Of all polyhedra the tetrahedron is the simplest element in three-dimensions (simplex). They can be used conveniently to partition computational domains involving complex geometries. Nowadays, tetrahedral meshes for complex geometries can be generated reliably employing commercial software packages.

The developed numerical method is part of an engineering method for predicting broadband noise within the flow about complex three-dimensional geometries. The broadband noise prediction method can be regarded as a three step method. In the first step the timeaveraged RANS-solution is obtained. In the second step turbulent aeroacoustic sources are obtained utilizing the time-averaged RANS-solution. In the third step the propagation of the aeroacoustic disturbances is simulated, employing the method described in this thesis, on unstructured tetrahedral meshes by solving the linearized Euler equations. In Blom et al. [16] the hybrid broadband noise prediction method is applied for the prediction of turbulence induced cavity-noise. In Snellen et al. [72] the hybrid broadband noise prediction method is applied to flow-induced noise around the A-pillar of an idealized car greenhouse.

### 1.4 Outline of Thesis

The linearized Euler equations are presented in chapter 2. The linearized Euler equations are derived from the more general Navier-Stokes equations. It is shown that the influence of viscosity and thermal conductivity on the propagation of acoustic information can be neglected, as a first approximation. In addition, it is shown that the amplitude of the acoustic perturbations are generally so small that it is justified to linearize the equations of motion about some mean flow. It is furthermore shown in this chapter, that, although, the linearized equations of motion can be obtained from the Euler equations in various ways, all but one of the resulting sets of linear equations can be rewritten in terms of any of the others, without introducing additional approximations.

In chapter 3 the discretization method, based on the DG method for the spatial discretization and the multi-stage low-storage Runge-Kutta time discretization method, is presented. The description is given for any desired order of accuracy of the method, in terms of the truncation error. In later chapters, where results, obtained for considered verification problems, are shown, the applied method has second-order accuracy in space and fourth-order accuracy in time. In chapter 3 also the treatment of domain boundary conditions and initial conditions are presented.

The developed method is to be applied for the simulation of the propagation of acoustic information. Algorithms applied for this purpose require assessment of their dispersion and dissipation errors, since these must be sufficiently low to accurately simulate the acoustic propagation. In chapter 4 an elaborate analysis of the wave-propagation properties of the DG method applied to an one-dimensional model problem is conducted. The analysis includes an assessment of the dispersion and dissipation error of the semi-discretization of the onedimensional model problem and a stability analysis of the fully-discrete algorithm. In the stability analysis the Euler explicit, Euler implicit and the multi-stage low-storage RungaKutta algorithms have been considered for the time discretization.

After developing an algorithm it is of utmost importance to verify the algorithm. Hereto, so-called, verification problems are considered, i.e. problems for which the analytical solution is known (or can be derived) and to with which numerically obtained results can be compared. Results obtained for two verification problems are presented in this thesis. In chapter 5 results are presented, obtained for the convection of a two dimensional Gaussian pulse, are presented. The Gaussian pulse problem is part of the ICASE/LaRC Workshop on Benchmark Problems in Computational Aeroacoustics (Atkins [3]). In chapter 5 the analyt-
ical solution for the problem will be derived and presented. Subsequently, numerical results are compared with the analytical solution.

The second verification problem considered is the acoustic radiation from a vibrating wall segment inside an infinite rectangular duct. In chapter 6 the analytical solution of the problem is presented. The numerical results obtained for the vibrating wall problem are presented in a separate chapter, chapter 7 .

Finally, concluding remarks and recommendations for future research are presented in chapter 8.

# ThE LINEARIZED EULER EQUATIONS 

### 2.1 Introduction

In this chapter the linearized Euler equations are derived from the more general Navier-Stokes equations. The Navier-Stokes equations can be used as mathematical model to describe a wide variety of fluid flows, including acoustic problems. A major drawback of a flow model based on the Navier-Stokes equations is that the partial differential equations are non-linear, complex and, in general, very difficult to solve.

Fortunately, (aero-) acoustics is almost exclusively concerned with sound propagation through fluids such as air and water that have only very small coefficients of viscosity and thermal conductivity. In addition, for the most part we are only interested in disturbances that are so small that their spatial gradients are never much larger than the disturbances themselves, the effects of viscosity and heat conduction can be neglected (Goldstein [34]); That is, when the propagation distances and frequencies involved are not too large. Under these assumptions the Navier-Stokes equations reduce to the Euler equations.

The amplitude of the disturbances, measured by the magnitude of the fluctuations, are usually weak (even for the loudest sounds); even weak enough to allow linearization of the equations of motion to obtain a first approximation of the equations governing propagation (Howe [41]). It is widely recognized that the propagation of sound can be described by the linearized Euler equations under the assumption that there is no significant feedback (or back-reaction) of the sound to the mean flow. Feedback is only to be expected when there is a resonator close to the flow field (Lighthill [54]). In the linearization process the possibility of representing feedback is lost. We therefore only consider (air-) flows in which feedback may be neglected.

In literature we encounter a variety of different sets of equations which are all designated as the linearized Euler equations, see for example [9], [10], [30], [31], [67]. This might be caused by the assumptions that are made, but also because the Euler equations can be written in different forms before introducing the linearization. The question rises whether linearizing these different forms results in the same set of equations. If so, we can truly speak about THE linearized Euler equations. First of all, the Euler equations can be written in terms of conservative or in terms of primitive variables. Furthermore the Euler equations, in terms of conservative variables, can be written in conservation form or in quasi-linear form, see also Fig.(2.1). Any of these expressions can be used as starting point for the linearization. Although we expect the respective results to be interchangeable, the proof can be cumbersome.

In this chapter we will show that the respective results are identical, which might be helpful for future users of the linearized Euler equations. It is noted, however, that some formulations are more beneficial for certain applications than others.


Figure 2.1: Schematic representation of different forms in which the Euler equations can appear.

In the next section we will start with the Navier-Stokes equations and show, by means of a dimension analysis, that the assumption of neglecting the attenuation of acoustic signals by viscosity and thermal conduction can be made. Subsequently we will show that the equations can be simplified even further by linearizing the equations with respect to a background flow. The actual linearization is performed in the third section of this chapter.

### 2.2 Aeroacoustics as an adiabatic and frictionless motion

Employing index notation and Cartesian coordinates, the Navier-Stokes equations in conservation form and with source terms, can be written as:

$$
\begin{align*}
& \frac{\partial \rho}{\partial t}+\frac{\partial}{\partial x_{j}}\left(\rho u_{j}\right)=S_{m},  \tag{2.1}\\
& \frac{\partial \rho u_{i}}{\partial t}+\frac{\partial}{\partial x_{j}}\left(\rho u_{j} u_{i}+p \delta_{i j}-\tau_{i j}\right)=S_{i}, \quad i=1,2,3,  \tag{2.2}\\
& \frac{\partial \rho E}{\partial t}+\frac{\partial}{\partial x_{j}}\left(\rho H u_{j}-\tau_{i j} u_{i}+q_{j}\right)=S_{e}, \tag{2.3}
\end{align*}
$$

where $\rho, u_{i}, p, E$ and $H$ are the density, the velocity components, the pressure, the total energy and the total enthalpy, respectively. Futhermore, $\delta_{i j}$ is the Kronecker delta function, $\vec{q}$ is the heat flux vector and $\tau_{i j}$ is the viscous stress tensor. The sources for the mass, momentum and energy equation are denoted by $S_{m}, S_{i}$ and $S_{e}$, respectively. In general $S_{m}=0$, since we consider a non-relativistic situation in which mass is conserved. However, the mass source
term $S_{m}$ can be used to represent a complex process (Rienstra \& Hirschberg [70]) such as, for example, the action of a pulsating sphere or of mass injection. The source term $S_{i}, i=1,2,3$ describes an external force field and $S_{e}$ describes the work done per unit time by the external force field and by external heat sources.

We will assume air to behave as a calorically perfect gas, i.e. the following relations are valid:

$$
\begin{align*}
& p=\rho R T,  \tag{2.4}\\
& e=c_{v} T, \tag{2.5}
\end{align*}
$$

where $T$ is the temperature, $e$ is the internal energy, $R$ is the specific gas constant, with $R=c_{p}-c_{v}$ and $c_{p}$ and $c_{v}$ are the specific heat at constant pressure and volume, respectively. The internal energy $(e)$ is related to the total energy $(E)$ and total enthalpy $(H)$ by:

$$
\begin{align*}
& E=e+\frac{1}{2} u_{k} u_{k}  \tag{2.6}\\
& H=E+\frac{p}{\rho} \tag{2.7}
\end{align*}
$$

Furthermore we use Fourier's law:

$$
\begin{equation*}
q_{i}=-\kappa \frac{\partial T}{\partial x_{i}} \tag{2.8}
\end{equation*}
$$

where $\kappa$ is the heat conduction coefficient. For a Newtonian fluid the viscous stress tensor is:

$$
\begin{equation*}
\tau_{i j}=\mu\left(\frac{\partial u_{i}}{\partial x_{j}}+\frac{\partial u_{j}}{\partial x_{i}}-\frac{2}{3} \delta_{i j} \frac{\partial u_{k}}{\partial x_{k}}\right) \tag{2.9}
\end{equation*}
$$

where $\mu$ is the dynamic viscosity coefficient, and we used Stokes' hypothesis. In the present study we assume that $\kappa$ and $\mu$ are constants.

### 2.2.1 From Navier-Stokes to Euler

There are four physical dimensions present in the equations: length, mass, time and temperature. By choosing a scaling parameter for each of the dimensions, we can write the equations in non-dimensional form. The Navier-Stokes equations contain parameters, like:

$$
\begin{equation*}
R, c_{v}, \mu, \kappa \tag{2.10}
\end{equation*}
$$

The boundary and initial conditions might contain parameters like:

$$
\begin{equation*}
f, \delta, T_{0}, U_{0}, \rho_{0}, L \tag{2.11}
\end{equation*}
$$

where $f$ and $\delta$ denote the frequency and amplitude of a specified wave and $T_{0}, U_{0}$ and $\rho_{0}$ are the temperature, velocity and density provided by the boundary or initial conditions. The length $L$ is related to the dimensions of the domain. From the parameters given in (2.10) and (2.11) we may choose four parameters $\wp_{i}$ such that

$$
\begin{equation*}
\wp_{1}{ }^{\alpha_{1}} \wp_{2}{ }^{\alpha_{2}} \wp_{3}{ }^{\alpha_{3}} \wp_{4}{ }^{\alpha_{4}}=1, \tag{2.12}
\end{equation*}
$$

has

$$
\begin{equation*}
\alpha_{1}=\alpha_{2}=\alpha_{3}=\alpha_{4}=0, \tag{2.13}
\end{equation*}
$$

as only solution. We choose the scaling parameters:

$$
\begin{equation*}
c_{v}, \rho_{0}, f, T_{0} \tag{2.14}
\end{equation*}
$$

The quantity:

$$
\begin{equation*}
\lambda=\frac{\sqrt{c_{v} T_{0}}}{f} \tag{2.15}
\end{equation*}
$$

has dimension of length and therefore represents a characteristic length scale, subsequently $\lambda f=\sqrt{c_{v} T_{0}}$ represents a characteristic velocity scale. For each quantity in the governing equations we can construct a scaling quantity of the form:

$$
\begin{equation*}
c_{v}{ }^{\alpha_{1}} \rho_{0}{ }^{\alpha_{2}} f^{\alpha_{3}} T_{0}{ }^{\alpha_{4}} \tag{2.16}
\end{equation*}
$$

such that dividing the quantity by the scaling quantity results in a dimensionless group. This procedure results in the following set of dimensionless equations:

$$
\begin{align*}
& \frac{\partial \bar{\rho}}{\partial \bar{t}}+\frac{\partial}{\partial \bar{x}_{j}}\left(\bar{\rho} \bar{u}_{j}\right)=\bar{S}_{m}  \tag{2.17}\\
& \frac{\partial \bar{\rho} \bar{u}_{i}}{\partial \bar{t}}+\frac{\partial}{\partial \bar{x}_{j}}\left(\bar{\rho} \bar{u}_{j} \bar{u}_{i}+\bar{p} \delta_{i j}-\bar{\tau}_{i j}\right)=\bar{S}_{i}  \tag{2.18}\\
& \frac{\partial \bar{\rho} \bar{E}}{\partial \bar{t}}+\frac{\partial}{\partial \bar{x}_{j}}\left(\bar{\rho} \bar{H} \bar{u}_{j}-\bar{\tau}_{i j} \bar{u}_{i}+\bar{q}_{j}\right)=\bar{S}_{e} \tag{2.19}
\end{align*}
$$

where $\bar{S}_{m}, \bar{S}_{i}$ and $\bar{S}_{e}$ are the dimensionless source terms:

$$
\begin{equation*}
\bar{S}_{m}=\frac{S_{m}}{\rho_{0} f}, \quad \bar{S}_{i}=\frac{S_{i}}{\rho_{0} \lambda f^{2}}, \quad \bar{S}_{e}=\frac{S_{e}}{\rho_{0} \lambda^{2} f^{3}} \tag{2.20}
\end{equation*}
$$

We can use Eqs.(2.8) and (2.9) to obtain:

$$
\begin{align*}
\bar{\tau}_{i j} & =\bar{\mu}\left(\frac{\partial \bar{u}_{i}}{\partial \bar{x}_{j}}+\frac{\partial \bar{u}_{j}}{\partial \bar{x}_{i}}-\frac{2}{3} \delta_{i j} \frac{\partial \bar{u}_{k}}{\partial \bar{x}_{k}}\right)  \tag{2.21}\\
\bar{q}_{j} & =-\bar{\kappa} \frac{\partial \bar{T}}{\partial \bar{x}_{j}} \tag{2.22}
\end{align*}
$$

where we introduced the two dimensionless parameter groups:

$$
\begin{align*}
\bar{\mu} & =\frac{\mu}{\rho_{0} \lambda^{2} f}  \tag{2.23}\\
\bar{\kappa} & =\frac{\kappa T_{0}}{\rho_{0} \lambda^{4} f^{3}} . \tag{2.24}
\end{align*}
$$

Upon identifying $\lambda$ as a characteristic length scale and $\lambda f$ a characteristic velocity scale, $\bar{\mu}^{-1}$ can be interpreted as an acoustic Reynolds number $\left(\operatorname{Re}_{a c}\right)$ :

$$
\begin{equation*}
\operatorname{Re}_{a c} \equiv \frac{\rho_{0} c_{v} T_{0}}{\mu f}=\frac{\rho_{0} \lambda^{2} f}{\mu}, \quad \Rightarrow \quad \bar{\mu}=\frac{1}{\operatorname{Re}_{a c}} \tag{2.25}
\end{equation*}
$$

Let us write $\bar{\kappa}$ as:

$$
\begin{equation*}
\bar{\kappa}=\frac{\mu}{\rho_{0} \lambda^{2} f} \cdot \frac{\kappa T_{0}}{\mu} \cdot \frac{1}{\lambda^{2} f^{2}} \tag{2.26}
\end{equation*}
$$

Then introducing theratio of specific heats:

$$
\begin{equation*}
\gamma=\frac{c_{p}}{c_{v}}, \tag{2.27}
\end{equation*}
$$

and the definition of the Prandtl number

$$
\begin{equation*}
\operatorname{Pr}=\frac{\mu c_{p}}{\kappa}, \tag{2.28}
\end{equation*}
$$

we can write for the dimensionless parameter group $\bar{\kappa}$ :

$$
\begin{equation*}
\bar{\kappa}=\frac{1}{\operatorname{Re}_{a c}} \cdot \frac{\gamma}{\operatorname{Pr}} . \tag{2.29}
\end{equation*}
$$

| quantity | value | dimension |
| :--- | :--- | :--- |
| $T_{0}$ | 293 | $K$ |
| $c_{v}$ | 717 | $m^{2} s^{-2} K^{-1}$ |
| $\nu=\mu / \rho$ | $1.5 .10^{-5}$ | $m^{2} s^{-2}$ |
| $\operatorname{Pr}$ | 0.72 |  |
| $\gamma$ | 1.4 |  |

TABLE 2.1: Characteristic values of some quantities of air at room temperature.
Employing table 2.1, we obtain $\frac{\gamma}{\operatorname{Pr}} \approx 1.94$ and

$$
\begin{equation*}
\operatorname{Re}_{a c}=\frac{\rho_{0} \lambda^{2} f}{\mu}=\frac{c_{v} T_{0}}{\nu f} \approx \frac{1 \cdot 4 \cdot 10^{10}}{f} . \tag{2.30}
\end{equation*}
$$

Obviously the acoustic Reynolds number will be large in the audible frequency range ( $f \in$ $[20 \mathrm{~Hz}, 20 \mathrm{kHz}])$ and for infrasound frequencies $(f<20)$. Even for a large part of the ultrasound frequencies (up to frequencies of several million $H z$ ) $\mathrm{Re}_{a c}$ will be large. If, for example, we look at $\operatorname{Re}_{a c}$ for frequencies ranging from $f=1 \mathrm{~Hz}$ to $f=100 \mathrm{MHz}$ (in medical ultrasound diagnostics frequencies up to 20 MHz are used), we have:

$$
\begin{equation*}
1.4 .10^{2}<\operatorname{Re}_{a c}<1.4 .10^{10} \tag{2.31}
\end{equation*}
$$

We may therefore, as a first approximation, neglect terms in the dimensionless equations that are multiplied by the parameter group $\mathrm{Re}_{a c}^{-1}$. The resulting equations are the dimensionless Euler equations:

$$
\begin{align*}
& \frac{\partial \bar{\rho}}{\partial \bar{t}}+\frac{\partial}{\partial \bar{x}_{j}}\left(\bar{\rho} \bar{u}_{j}\right)=\bar{S}_{m}  \tag{2.32}\\
& \frac{\partial \bar{\rho} \bar{u}_{i}}{\partial \bar{t}}+\frac{\partial}{\partial \bar{x}_{j}}\left(\bar{\rho} \bar{u}_{j} \bar{u}_{i}+\bar{p} \delta_{i j}\right)=\bar{S}_{i}  \tag{2.33}\\
& \frac{\partial \bar{\rho} \bar{E}}{\partial \bar{t}}+\frac{\partial}{\partial \bar{x}_{j}}\left(\bar{\rho} \bar{H} \bar{u}_{j}\right)=\bar{S}_{e} \tag{2.34}
\end{align*}
$$

We may conclude that, for frequencies which are not too large, acoustics can be described as a frictionless motion where heat conduction may be neglected. Since $\operatorname{Re}_{a c}$ decreases with increasing frequency the attenuation of sound waves increases, in general, with increasing frequency. When the distance over which the acoustic waves propagate becomes larger and larger, the influence of viscous dissipation and thermal conduction becomes larger as well. This explains why we hear the lower frequencies of an airplane more and more accentuated as it flies from near the observation point (e.g. airport) away to large distances (say 10 km ) ([70]). We must therefore keep in mind that the Euler equations only give a good approximation when the frequencies and the propagation distances involved are not too large.

Note that in monitoring earthquakes frequencies as low as $f=0.001 \mathrm{~Hz}$ are used. For $f=$ 0.001 Hz the acoustic Reynolds number is very large, viz. $\approx 1.4 .10^{13}$, and the influence of viscous dissipation and thermal conduction may be negelected. However, when considering extremely low frequencies, of the order of $g / c$ ( $\approx 0.029$ in air at atmospheric conditions), where $g$ is the acceleration due to gravity and $c$ is the speed of sound, the influence of gravity, which in general has a negligible influence, can no longer be neglected [63].

In the presence of walls the viscous dissipation and thermal conduction will result in a significantly larger attenuation over shorter distances. The amplitude of a plane wave traveling along a circular tube of radius $r$ will decrease with the distance $x$ along the tube following an exponential factor $e^{-\alpha x}$, where the damping factor $\alpha$ is given, at reasonably high frequencies, by ([63]):

$$
\begin{equation*}
\alpha=\frac{4 \sqrt{\pi f \nu}}{r c_{0}}\left(1+\frac{\gamma-1}{\sqrt{\operatorname{Pr}}}\right) . \tag{2.35}
\end{equation*}
$$

For air at atmospheric conditions the speed of sound is $c_{0}=344 \mathrm{~ms}^{-1}$, so for example, for a musical instrument of radius 2.5 cm , we obtain $\alpha=4.7 .10^{-3} \sqrt{f}$. For $f=1 \mathrm{kHz}$ we then have $\alpha \approx 0.148$, which implies that the amplitude of the signal will be halved in about 4.7 $m$.

Note that often when conducting a dimension analysis the speed of sound is used to scale velocities. The speed of sound $\left(c_{0}\right)$ is defined as:

$$
\begin{equation*}
c_{0}^{2}=\left(\frac{\partial p}{\partial \rho}\right)_{s} \tag{2.36}
\end{equation*}
$$

where the subscript $s$ means: taking the partial derivative at constant entropy. Employing thermodynamic and Maxwell relations we obtain for the speed of sound for a calorically perfect gas:

$$
\begin{equation*}
c_{0}^{2}=\gamma\left(\frac{\partial p}{\partial \rho}\right)_{T} \tag{2.37}
\end{equation*}
$$

For a perfect gas we then obtain the well-known relation: $c_{0}^{2}=\gamma R T$. The relation between the characteristic velocity scale, $\lambda f$, used to scale the velocities in this section and the speed of sound is now given by ${ }^{1}$ :

$$
\begin{equation*}
\lambda f=\frac{c_{0}}{\sqrt{\gamma(\gamma-1)}} \sqrt{\frac{T_{0}}{T}}, \tag{2.38}
\end{equation*}
$$

[^1]where $\gamma$ is constant. In later chapters the equations are made dimensionless employing the speed of sound rather than $\lambda f$, as one of the scaling quantities.

### 2.2.2 From Euler to linearized Euler

Sound involves a large range of power levels. For example when whispering, we produce about $10^{-10}$ Watt, when shouting about $10^{-5}$ Watt and a jet airplane at take-off produces about $10^{5}$ Watt of acoustic power [29]. Because of this large range of power levels and because our ear has a logarithmic type of sensitivity, we usually use the decibel scale to measure sound levels ([70]). The Sound Pressure Level (SPL) is given, in decibel ( $d B$ ), by:

$$
\begin{equation*}
\mathrm{SPL}=20 \log _{10}\left(\frac{p_{r m s}^{\prime}}{p_{r e f}}\right) \tag{2.39}
\end{equation*}
$$

where $p_{r m s}^{\prime}$ is the root mean square of the acoustic pressure fluctuations $p^{\prime}$, and where $p_{r e f}=$ $2.10^{-5} \mathrm{~Pa}$ in air. This reference pressure corresponds to the threshold of hearing at 1 kHz for a typical human ear. Humans, most humans at least, are able to endure 140 dB for a short period of time without risk of permanent ear damage ([70]), this corresponds in air to pressure fluctuations of $p_{r m s}^{\prime}=200 \mathrm{~Pa}$.

If we neglect all the source terms in Eq.(2.34), aeroacoustics consists of an isentropic motion ( $s=$ constant along particle paths, assuming that there are no shocks present in the flow). For a perfect gas, which we assume air to be, we may write the first law of thermodynamics as $d q=c_{v} d T+p d v$, with $v$ the specific volume. In case of an adiabatic process $d q=0$ we obtain:

$$
\begin{equation*}
c_{v} d T=-p d\left(\frac{1}{\rho}\right)=\frac{p}{\rho^{2}} d \rho \tag{2.40}
\end{equation*}
$$

From the equation of state, Eq.(2.4), we obtain:

$$
\begin{equation*}
d T=\frac{1}{R}\left(\frac{1}{\rho} d p-\frac{p}{\rho^{2}} d \rho\right) \tag{2.41}
\end{equation*}
$$

Combination of equations (2.40) and (2.41) gives:

$$
\begin{equation*}
\frac{c_{v}}{R}\left(d p-\frac{p}{\rho} d \rho\right)=\frac{p}{\rho^{2}} d \rho \tag{2.42}
\end{equation*}
$$

With $R=c_{p}-c_{v}$ and $\gamma=\frac{c_{p}}{c_{v}}$ we finally obtain:

$$
\begin{equation*}
\frac{d p}{d \rho}=\gamma \frac{p}{\rho} \tag{2.43}
\end{equation*}
$$

Therefore, at atmospheric pressure $p_{0} \approx 10^{5} \mathrm{~Pa}$, the relative density fluctuations are, for a decibel level of 140 dB :

$$
\begin{equation*}
\frac{\rho^{\prime}}{\rho_{0}} \approx \frac{p^{\prime}}{\gamma p_{0}}<1.35 .10^{-3} \tag{2.44}
\end{equation*}
$$

If, for example, we allow linearization when the maximum disturbance is not larger than $10 \%$ of the undisturbed value, we get:

$$
p_{\max }^{\prime}=0.1 p_{0}=10^{4} P a,
$$

$$
\begin{align*}
& \rho_{\max }^{\prime}=\frac{0.1}{\gamma} \rho_{0} \Rightarrow \frac{\rho^{\prime}}{\rho_{0}}=0.071 \\
& \mathrm{SPL}_{\max }=174 \mathrm{~dB} \tag{2.45}
\end{align*}
$$

This example demonstrates that even for very high sound pressure levels, the pressure disturbances are small enough to allow linearization as a first approximation.

### 2.3 Linearizing the Euler equations

We have shown that we may, as a first approximation, linearize the Euler equations to obtain a set of equations, called the linearized Euler equations (LEE), that describe (aero-) acoustic phenomena. In literature we encounter a variety of sets of equations, that are all called the linearized Euler equations. This might be caused by the Euler equations appearing in different forms before introducing the linearization. The Euler equations can be written in terms of conservative or primitive variables, $\mathbf{u}^{\prime}$ and $\mathbf{q}^{\prime}$, respectively. Furthermore the Euler equations, in terms of conservative variables, can be written in conservation form or in quasilinear form. The question rises whether linearizing these different forms results in the same set of equations. If so, we can truly speak about THE linearized Euler equations.


Figure 2.2: Schematic representation of three different forms in which the Euler equations can appear and how linearization of these equations leads to different formulations of the linear equations. The connections denoted by the dotted curves are to be proven in sections 2.3.2 and 2.3.3.

Fig.(2.1) gives a schematic representation of the three mentioned forms in which the Euler equations can appear and how these lead, by linearization, to different forms in which the linearized equations can appear. In Fig.(2.1) we use acronyms for the different forms of the Euler equation:

ECC: Euler equations in Conservation form for Conservative variables
EQC: Euler equations in Quasi-linear form for Conservative variables
EP: Euler equations for Primative variables
When linearizing we assume the aeroacoustic perturbations ( $\rho^{\prime}, u_{i}^{\prime}, p^{\prime}$ ) to be small compared to the mean flow properties $\left(\rho_{0}, u_{i 0}, p_{0}\right)$. So, we assume:

$$
\begin{equation*}
\frac{\left|q^{\prime}\right|}{\left|q_{0}\right|}=\mathcal{O}(\varepsilon), \quad \varepsilon \ll 1, \tag{2.46}
\end{equation*}
$$

where $q$ is either $\rho, u_{i}$ or $p$. Furthermore it is assumed that the mean flow quantities satisfy the Euler equations. We will substitute $\rho=\rho_{0}+\rho^{\prime}, u_{i}=u_{i 0}+u^{\prime}{ }_{i}$ and $p=p_{0}+p^{\prime}$ into the equations and neglect terms of the order $\mathcal{O}\left(\varepsilon^{2}\right)$. For the source terms we assume:

$$
\begin{equation*}
\left|\mathbf{S}^{\prime}\right|=\left|\mathbf{S}-\mathbf{S}_{0}\right|=\left|\mathbf{S}_{0}\right| \mathcal{O}(\varepsilon), \tag{2.47}
\end{equation*}
$$

and we substitute $S=S_{0}+S^{\prime}$, where $S$ represents here either the source term for the continuity, momentum or energy equation, in the equations.

In section 2.3.1 we will derive formulations 1 to 3 for the linearized equations, starting from the Euler equations in conservation form for conservative variables (ECC). The first formulation is obtained directly (there are no manipulations of the result involved) from linearizing the ECC. The second formulation is obtained after some manipulation and rearranging of the first result (no terms are neglected). The last, and most popular, formulation is obtained after further manipulation and rearranging of the second result. In this formulation, formulation 3, the LEE are written in terms of the primitive perturbation variables.

In section 2.3.2 we linearize the EQC to obtain formulation 4 and we will show that relation 4 is identical to formulation 1 . In section 2.3 .3 we obtain formulation 5 when starting the linearization process from the EP. In section 2.3 .3 we will also show that formulations 3 and 5 as well as 4 and 5 are identical.

### 2.3.1 Linearizing the Euler equations in conservation form for conservative variables

The Euler equations are presented in Eqs.(2.32 to (2.34) in dimensionless form. For the sake of completeness we will present them here again, but now without the overbar notation.

$$
\begin{align*}
& \frac{\partial \rho}{\partial t}+\frac{\partial}{\partial x_{j}}\left(\rho u_{j}\right)=S_{m}  \tag{2.48}\\
& \frac{\partial \rho u_{i}}{\partial t}+\frac{\partial}{\partial x_{j}}\left(\rho u_{j} u_{i}+p \delta_{i j}\right)=S_{i}  \tag{2.49}\\
& \frac{\partial \rho E}{\partial t}+\frac{\partial}{\partial x_{j}}\left(\rho H u_{j}\right)=S_{e} \tag{2.50}
\end{align*}
$$

The ECC in vector notation are given by:

$$
\begin{equation*}
\frac{\partial \mathbf{U}}{\partial t}+\frac{\partial}{\partial x_{j}} \mathbf{F}_{j}(\mathbf{U})=\mathbf{S}(\mathbf{U}) \tag{2.51}
\end{equation*}
$$

where it is noted that $\mathbf{S}$ can be a function of both $\mathbf{x}$ and $t$ as well. Furthermore:

$$
\mathbf{U}=\left(\begin{array}{c}
\rho  \tag{2.52}\\
\rho u \\
\rho v \\
\rho w \\
\rho E
\end{array}\right), \quad \mathbf{F}_{j}=\left(\begin{array}{c}
\rho u_{j} \\
\rho u u_{j}+\delta_{1 j} p \\
\rho v u_{j}+\delta_{2 j} p \\
\rho w u_{j}+\delta_{3 j} p \\
\rho H u_{j}
\end{array}\right), \quad \mathbf{S}=\left(\begin{array}{c}
S_{m} \\
S_{1} \\
S_{2} \\
S_{3} \\
S_{e}
\end{array}\right)
$$

and $\rho E=\frac{1}{\gamma-1} p+\frac{1}{2} \rho u_{k} u_{k}, \rho H=\frac{\gamma}{\gamma-1} p+\frac{1}{2} \rho u_{k} u_{k}$.

## Formulation 1

## Formulation 1a

Upon introducing the following notation for the conservative variables:

$$
\mathbf{U}=\left(\begin{array}{c}
\rho  \tag{2.53}\\
\rho u \\
\rho v \\
\rho w \\
\rho E
\end{array}\right)=\left(\begin{array}{c}
w_{1} \\
w_{2} \\
w_{3} \\
w_{4} \\
w_{5}
\end{array}\right)
$$

the fluxes $\mathbf{F}_{j}$ of Eq.(2.52) can be rewritten in terms of the $w_{k}$ 's to obtain:

$$
\left.\left.\mathbf{F}_{j}(\mathbf{U})=\left(\begin{array}{c}
w_{1+j}  \tag{2.54}\\
\frac{w_{2} w_{j+1}}{w_{1}}+\delta_{1 j}(\gamma-1)\left[w_{5}-\frac{1}{2 w_{1}}\left(w_{2}^{2}+w_{3}^{2}+w_{4}^{2}\right)\right. \\
\frac{w_{3} w_{j+1}}{w_{1}}+\delta_{2 j}(\gamma-1) \\
\frac{w_{4} w_{j+1}}{w_{1}}+\delta_{3 j}(\gamma-1) \\
\frac{w_{j+1}}{w_{1}}\left[\gamma w_{5}-\frac{1}{2 w_{1}}\left(w_{2}^{2}+w_{3}^{2}+w_{4}^{2}\right)\right. \\
2 w_{5}-\frac{1}{2 w_{1}}\left(w_{2}^{2}+w_{3}^{2}+w_{2}^{2}\right)
\end{array}\right] . w_{3}^{2}+w_{4}^{2}\right)\right] .
$$

Let us assume that the mean flow quantities $\mathbf{U}_{0}=\mathbf{W}_{0}=\left(\rho_{0}, \rho_{0} u_{i 0}, \rho_{0} E_{0}\right)$ satisfy the Euler equations, including source term:

$$
\begin{equation*}
\frac{\partial \mathbf{U}_{0}}{\partial t}+\frac{\partial}{\partial x_{j}} \mathbf{F}_{j}\left(\mathbf{U}_{0}\right)=\mathbf{S}\left(\mathbf{U}_{0}\right), \quad \mathbf{S}\left(\mathbf{U}_{0}\right)=\mathbf{S}_{0} \tag{2.55}
\end{equation*}
$$

In the current formulation it is assumed that:

$$
\begin{equation*}
\left|w_{k}^{\prime}\right|=\left|w_{k 0}\right| \mathcal{O}(\epsilon) \tag{2.56}
\end{equation*}
$$

such that we can split any perturbed solution $\mathbf{U}$ in Eq.(2.51) as:

$$
\begin{equation*}
\mathbf{U}=\mathbf{U}_{0}+\mathbf{W}^{\prime} \quad \text { or } \quad \mathbf{U}=\mathbf{W}_{0}+\mathbf{W}^{\prime}, \quad \mathbf{U}_{0}=\mathbf{W}_{0} \tag{2.57}
\end{equation*}
$$

Substitution of $w_{k}=w_{k 0}+w_{k}^{\prime}, k=1,2, \ldots, 5, S_{m}=S_{m 0}+S_{m}^{\prime}, S_{i}=S_{i 0}+S_{i}^{\prime}$, $i=1,2,3$ and $S_{e}=S_{e 0}+S_{e}^{\prime}$ into Eq.(2.51), with $\mathbf{U}$ given by Eq.(2.53) and $\mathbf{F}_{j}$ by Eq.(2.54) results in the following set of equations for the perturbation variables $w_{k}^{\prime}(\mathcal{O}(\epsilon))$ :

$$
\begin{align*}
& \frac{\partial}{\partial t}\left(\begin{array}{c}
w_{1}^{\prime} \\
w_{2}^{\prime} \\
w_{3}^{\prime} \\
w_{4}^{\prime} \\
w_{5}^{\prime}
\end{array}\right)+\frac{\partial}{\partial x_{j}}\left\{\left(\begin{array}{c}
w_{j+1}^{\prime} \\
\frac{w_{j+10} w_{2}^{\prime}+w_{20} w_{j+1}^{\prime}}{w_{10}}-\frac{w_{j+10} w_{20} w_{1}^{\prime}}{w_{10}^{2}} \\
\frac{w_{j+10} w_{3}^{\prime}+w_{30} w_{j+1}^{\prime}}{w_{10}}-\frac{w_{j+1} w_{30} w_{1}^{\prime}}{w_{10}^{2}} \\
\frac{w_{j+10} w_{4}^{\prime}+w_{40} w_{j+1}^{\prime}}{w_{10}}-\frac{w_{j+10} w_{40} w_{1}^{\prime}}{w_{10}^{2}} \\
\left(\frac{w_{j+1}^{\prime}}{w_{10}}-\frac{w_{j+10} w_{1}^{\prime}}{w_{10}^{2}}\right)\left[\gamma w_{50}-\frac{\gamma-1}{2 w_{10}} w_{k 0} w_{k 0}\right]
\end{array}\right)\right. \\
& \left.+\left(\begin{array}{c}
\delta_{1 j}(\gamma-1)\left[\begin{array}{c}
w_{5}^{\prime}-\frac{w_{k 0} w_{k}^{\prime}}{2 w_{10}}+\frac{w_{k 0} w_{k 0} w_{1}^{\prime}}{2 w_{10}^{2}} \\
\delta_{2 j}(\gamma-1)
\end{array}\right] w_{5}^{\prime}-\frac{w_{k 0} w_{k}^{\prime}}{2 w_{10}}+\frac{w_{k 0} w_{k 0} w_{1}^{\prime}}{2 w_{10}^{2}} \\
\delta_{3 j}(\gamma-1)\left[w_{5}^{\prime}-\frac{w_{k 0} w_{k}^{\prime}}{2 w_{10}}+\frac{w_{k 0} w_{k 0} w_{1}^{\prime}}{2 w_{10}^{2}}\right] \\
\frac{w_{j+10}}{w_{10}}\left[\gamma w_{5}^{\prime}-\frac{\gamma-1}{2 w_{10}} w_{k 0} w_{k}^{\prime}+\frac{\gamma-1}{2 w_{10}^{2}} w_{k 0} w_{k 0} w_{1}^{\prime}\right]
\end{array}\right)\right\} . \tag{2.58}
\end{align*}
$$

The set of equations given by Eq.(2.58) presents the LEE in the most generic form.

## Formulation $1 b$

Assuming that the mean flow quantities satisfy the Euler equations including source term Eq.(2.56), substitution of $\rho=\rho_{0}+\rho^{\prime}, u_{i}=u_{i 0}+u^{\prime}{ }_{i}, p=p_{0}+p^{\prime}$ and $S=S_{0}+S^{\prime}$, for the individual sources, into Eq.(2.51), neglecting terms of order $\mathcal{O}\left(\varepsilon^{2}\right)$ and using Eq.(2.56), results in:

$$
\begin{equation*}
\frac{\partial \mathbf{U}^{\prime}}{\partial t}+\frac{\partial \mathbf{F}_{j}^{\prime}}{\partial x_{j}}=\mathbf{S}^{\prime} \tag{2.59}
\end{equation*}
$$

where

$$
\begin{align*}
& \mathbf{U}^{\prime}=\left(\begin{array}{c}
\rho^{\prime} \\
\rho_{0} u^{\prime}+u_{0} \rho^{\prime} \\
\rho_{0} v^{\prime}+v_{0} \rho^{\prime} \\
\rho_{0} w^{\prime}+w_{0} \rho^{\prime} \\
\rho_{0} u_{k 0} u^{\prime}{ }_{k}+\frac{1}{2}\left|\vec{u}_{0}\right|^{2} \rho^{\prime}+\frac{1}{\gamma-1} p^{\prime}
\end{array}\right), \quad \mathbf{S}^{\prime}=\left(\begin{array}{c}
S_{m}^{\prime} \\
S_{1}^{\prime} \\
S_{2}^{\prime} \\
S_{3}^{\prime} \\
S_{e}^{\prime}
\end{array}\right),  \tag{2.60}\\
& \rho_{0} u_{j}^{\prime}+u_{j 0} \rho^{\prime}  \tag{2.61}\\
& \mathbf{F}_{j}^{\prime}=\left(\begin{array}{c}
\rho_{0} u_{0} u^{\prime}{ }_{j}+\rho_{0} u_{j 0} u^{\prime}+u_{0} u_{j 0} \rho^{\prime}+\delta_{1 j} p^{\prime} \\
\rho_{0} v_{0} u^{\prime}{ }_{j}+\rho_{0} u_{j 0} v^{\prime}+v_{0} u_{j 0} \rho^{\prime}+\delta_{2 j} p^{\prime} \\
\rho_{0} w_{0} u^{\prime}{ }_{j}+\rho_{0} u_{j 0} w^{\prime}+w_{0} u_{j 0} \rho^{\prime}+\delta_{3 j} p^{\prime} \\
\rho_{0} u_{j 0} u_{k 0} u^{\prime}{ }_{k}+\frac{1}{2} u_{k 0} u_{k 0}\left(\rho_{0} u^{\prime}{ }_{j}+u_{j 0} \rho^{\prime}\right)+\frac{\gamma}{\gamma-1}\left(p_{0} u^{\prime}{ }_{j}+u_{j 0} p^{\prime}\right)
\end{array}\right) .
\end{align*}
$$

Note that $\mathbf{U}^{\prime}$ is the $\mathcal{O}(\epsilon)$ part of $\mathbf{U}$ expanded in terms of the primitive variables $\rho, u_{i}$ and $p$.
The set of equations given by Eq.(2.59) could also have been obtained as follows:
Let us assume we have a solution $\mathbf{U}_{0}$ satisfying the Euler equations of Eq.(2.56), such that we can split any perturbed solution $\mathbf{U}$ :

$$
\begin{equation*}
\mathbf{U}=\mathbf{U}_{0}+\mathbf{U}^{\prime} \Rightarrow \mathbf{U}^{\prime}=\mathbf{U}-\mathbf{U}_{0} \tag{2.62}
\end{equation*}
$$

Substitution of the perturbed solution into Eq.(2.51) yields:

$$
\begin{equation*}
\frac{\partial}{\partial t}\left(\mathbf{U}_{0}+\mathbf{U}^{\prime}\right)+\frac{\partial}{\partial x_{j}}\left(\mathbf{F}_{j}\left(\mathbf{U}_{0}+\mathbf{U}^{\prime}\right)\right)=\mathbf{S} \tag{2.63}
\end{equation*}
$$

which is a non-linear equation. We can linearize equation (2.63) when

$$
\begin{equation*}
\left|\mathbf{S}^{\prime}\right|=\left|\mathbf{S}-\mathbf{S}_{0}\right|=\left|\mathbf{S}_{0}\right| \mathcal{O}(\epsilon), \quad \epsilon \ll 1 . \tag{2.64}
\end{equation*}
$$

Upon using Taylor series expansion of the flux vector we obtain:

$$
\begin{equation*}
\frac{\partial}{\partial t}\left(\mathbf{U}_{0}+\mathbf{U}^{\prime}\right)+\frac{\partial}{\partial x_{j}}\left(\mathbf{F}_{j}\left(\mathbf{U}_{0}\right)+\frac{d \mathbf{F}_{j}}{d \mathbf{U}}\left(\mathbf{U}_{0}\right) \mathbf{U}^{\prime}\right)=\mathbf{S}_{0}+\mathbf{S}^{\prime}+\mathcal{O}\left(\epsilon^{2}\right) \tag{2.65}
\end{equation*}
$$

from which we obtain the equation for the $\mathcal{O}(\epsilon)$ perturbation:

$$
\begin{equation*}
\frac{\partial \mathbf{U}^{\prime}}{\partial t}+\frac{\partial}{\partial x_{j}}\left(B_{j}\left(\mathbf{U}_{0}\right) \mathbf{U}^{\prime}\right)=\mathbf{S}^{\prime} \tag{2.66}
\end{equation*}
$$

The matrices $B_{j 0}=B_{j}\left(\mathbf{U}_{0}\right)=\frac{d \mathbf{F}_{j}}{d \mathbf{U}}\left(\mathbf{U}_{0}\right)$ are derived and presented in appendix A.1. It can be shown that working out the product $B_{j 0} \mathbf{U}^{\prime}$ results in $B_{j 0} \mathbf{U}^{\prime}=\mathbf{F}_{j}^{\prime}$. In Eq.(2.65) we have written $\mathcal{O}\left(\epsilon^{2}\right)$ as a short-hand notation, where we actually mean that all five equations have a term of order $\mathcal{O}\left(\epsilon^{2}\right)$.

Eq.(2.66) can also be written as:

$$
\begin{equation*}
\frac{\partial \mathbf{U}^{\prime}}{\partial t}+B_{j 0} \frac{\partial \mathbf{U}^{\prime}}{\partial x_{j}}=\mathbf{S}^{\prime}-\frac{\partial B_{j 0}}{\partial x_{j}} \mathbf{U}^{\prime} \tag{2.67}
\end{equation*}
$$

Redonnet et al. [67], for example, use the set of equations given by Eq.(2.59) as the LEE. Redonnet et al. motivate their choice by emphasizing the physical generality of Eq.(2.59), which might lead to a better understanding of their numerical treatment. As source term Redonnet et al. use a vector which closely resembles the perturbation vector $\mathbf{U}^{\prime}$ (Eq.(2.60)), the difference being that the primitive perturbation variables, $\rho^{\prime}, u_{i}^{\prime}, p^{\prime}$, have been replaced by arbitrary (though known) acoustic forcing functions $\rho_{s}^{\prime}, u_{i s}^{\prime}, p_{s}^{\prime}$.

## Relation between formulation 1a and 1b

Upon writing $\mathbf{W}^{\prime}$ in terms of the primitive variables the difference between formulations 1 a and $b$ becomes clear immediately:

$$
\begin{align*}
w_{1}^{\prime} & =\rho^{\prime}  \tag{2.68}\\
w_{i}^{\prime} & =\left(\rho_{0}+\rho^{\prime}\right)\left(u_{i 0}+u_{i}^{\prime}\right)-\rho_{0} u_{i 0}=\rho_{0} u_{i}^{\prime}+u_{i 0} \rho^{\prime}, \quad i=1,2,3  \tag{2.69}\\
w_{5}^{\prime} & =\frac{1}{\gamma-1}\left(p_{0}+p^{\prime}\right)+\frac{1}{2}\left(\rho_{0}+\rho^{\prime}\right)\left(u_{k 0}+u_{k}^{\prime}\right)\left(u_{k 0}+u_{k}^{\prime}\right)-\rho_{0} E_{0} \\
& =\frac{1}{\gamma-1} p^{\prime}+\rho_{0} u_{k 0} u_{k}^{\prime}+u_{k 0} u_{k}^{\prime} \rho^{\prime}+\frac{1}{2}\left(u_{k 0} u_{k 0} \rho^{\prime}+\rho_{0} u_{k}^{\prime} u_{k}^{\prime} \rho^{\prime} u_{k}^{\prime} u_{k}^{\prime}\right) . \tag{2.70}
\end{align*}
$$

Hence, in terms of the primitive perturbation variables $\mathbf{U}^{\prime}$ we have:

$$
\begin{equation*}
\mathbf{W}^{\prime}=\mathbf{U}^{\prime}+\mathcal{O}(\varepsilon) \tag{2.71}
\end{equation*}
$$

and the difference between Eq.(2.59) and Eq.(2.58) is of the order $\mathcal{O}\left(\varepsilon^{2}\right)$.

## Formulation 2

A related formulation can be obtained from the above set, Eq.(2.59), upon employing both the continuity and the momentum equations for the mean flow and the perturbation. Substitution of these relations results, after some rearranging, in the following set of equations:

$$
\begin{equation*}
\frac{\partial \tilde{\mathbf{U}}^{\prime}}{\partial t}+\frac{\partial \tilde{\mathbf{F}}_{j}^{\prime}}{\partial x_{j}}+\tilde{\mathbf{H}}^{\prime}=\tilde{\mathbf{S}}^{\prime} \tag{2.72}
\end{equation*}
$$

where

$$
\begin{align*}
& \tilde{\mathbf{U}}^{\prime}=\left(\begin{array}{c}
\rho^{\prime} \\
\rho_{0} u^{\prime} \\
\rho_{0} v^{\prime} \\
\rho_{0} w^{\prime} \\
p^{\prime}
\end{array}\right), \quad \tilde{\mathbf{S}}^{\prime}=\left(\begin{array}{c}
S_{m}^{\prime} \\
S_{1}^{\prime}-u_{0} S_{m}^{\prime}-\frac{\rho^{\prime}}{\rho_{0}}\left(S_{10}-u_{0} S_{m 0}\right) \\
S_{2}^{\prime}-v_{0} S_{m}^{\prime}-\frac{\rho_{0}}{\rho_{0}}\left(S_{20}-v_{0} S_{m 0}\right) \\
S_{3}^{\prime}-w_{0} S_{m}^{\prime}-\frac{\rho}{\rho_{0}}\left(S_{30}-w_{0} S_{m 0}\right) \\
S_{e}^{\prime}-u_{k}^{\prime} S_{k 0}-u_{k 0} S_{k}^{\prime}+\frac{1}{2}\left|\vec{u}_{0}\right|^{2} S_{m}^{\prime}+u_{k 0} u_{k}^{\prime} S_{m 0}
\end{array}\right),  \tag{2.73}\\
& \tilde{\mathbf{F}}_{j}^{\prime}=\left(\begin{array}{c}
\rho_{0} u^{\prime}{ }_{j}+u_{j o} \rho^{\prime} \\
\rho_{0} u_{j 0} u^{\prime}+\delta_{1 j} p^{\prime} \\
\rho_{0} u_{j 0} v^{\prime}+\delta_{2 j} p^{\prime} \\
\rho_{0} u_{j o} w^{\prime}+\delta_{3 j} p^{\prime} \\
u_{j 0} p^{\prime}+\gamma p_{0} u_{j}^{\prime}
\end{array}\right), \quad \tilde{\mathbf{H}}^{\prime}=\left(\begin{array}{c}
0 \\
\rho_{0} u_{j}^{\prime} \frac{\partial u_{0}}{\partial x_{j}}-\frac{\rho^{\prime}}{\rho_{0}} \frac{\partial p_{0}}{\partial x} \\
\rho_{0} u_{j}^{\prime} \frac{\partial v_{0}}{\partial x_{j}}-\frac{\rho_{0}^{\prime}}{\rho_{0}} \frac{\partial p_{0}}{\partial y} \\
\rho_{0} u_{j}^{\prime} \frac{\partial w_{0}}{\partial x_{j}}-\rho^{\prime} \\
\rho_{0} \frac{\partial p_{0}}{\partial z} \\
(\gamma-1)\left(p^{\prime} \frac{p^{\prime}}{\partial x_{j}}-u_{j}^{\prime} \frac{p_{0}}{\partial x_{j}}\right)
\end{array}\right) . \tag{2.74}
\end{align*}
$$

The vector $\tilde{\mathbf{H}}^{\prime}$ can be seen as a source term, related to gradients in the background flow. Bailly et al. ([9] and [10]) use the above set of equations as the LEE together with the assumption that mean flow is steady. Under this assumption the second, third and fourth term in $\tilde{\mathbf{H}}^{\prime}$ can be written as $\left(\rho_{0} u_{j}^{\prime}+u_{i 0} \rho^{\prime}\right) \frac{\partial u_{0}}{\partial x_{j}}$, with $i=1,2$ and 3 , respectively.

In [10], Bailly et al. consider a sheared mean flow and they set the sources $\tilde{\mathbf{S}}_{1}^{\prime}$ and $\tilde{\mathbf{S}}_{5}^{\prime}$ to zero. Furthermore, Bailly et al. replace $\tilde{\mathbf{S}}_{i}^{\prime}, i=1,2,3$ by the sources $S_{i}^{f}-\overline{S_{i}^{f}}$, where $S_{i}^{f}=-\frac{\partial}{\partial x_{j}}\left(\rho_{0} u_{i}^{\prime} u_{j}^{\prime}\right)$ are non-linear in the velocity fluctuations. As is generally known (and as will be shown in section 2.4), the LEE can be cast into a wave equation. The wave equation which can be obtained from the LEE combined with the sources chosen by Bailly et al. [9], [10], can be interpreted as a simplified Lilley's equation for an unilateral sheared mean flow, as is shown in [10]. Lilley's equation is equivalent to the well known Lighthill's equation (Lighthill's equation will be briefly describes in section 2.4), however in Lilley's equation some of the interaction between sound field and the mean flow has been moved from the source term to the wave-operator part of the equation ([34]). More about Lilley's equation can be found in, amongst others, [34]. Note that also the famous wave equation of Lighthill has a source term comprised of these non-linear velocity fluctuations.

## Formulation 3

In the most popular formulation of the LEE the unknowns are the primitive perturbation variables: $\mathbf{q}^{\prime}=\left(\rho^{\prime}, u^{\prime}, v^{\prime}, w^{\prime}, p^{\prime}\right)^{T}$. The linearized Euler equations in terms of the primitive perturbation variables can be obtained from Eq.(2.72) by further manipulation. Hereto the
continuity and momentum equations of both the mean flow and the perturbation variables have to be used. The system of equations can be written as:

$$
\begin{equation*}
\frac{\partial \mathbf{q}^{\prime}}{\partial t}+A_{j 0} \frac{\partial \mathbf{q}^{\prime}}{\partial x_{j}}+A_{j}^{\prime} \frac{\partial \mathbf{q}_{0}}{\partial x_{j}}=\mathbf{Q}^{\prime}, \tag{2.75}
\end{equation*}
$$

where $\mathbf{q}_{0}=\left(\rho_{0}, u_{0}, v_{0}, w_{0}, p_{0}\right)^{T}$ and where the matrices $A_{j 0}$ and $A_{j}^{\prime}$ are given by:

$$
\begin{align*}
& A_{j 0}=\left(\begin{array}{ccccc}
u_{j 0} & \delta_{1 j} \rho_{0} & \delta_{2 j} \rho_{0} & \delta_{3 j} \rho_{0} & 0 \\
0 & u_{j 0} & 0 & 0 & \frac{\delta_{1 j}}{\rho_{0}} \\
0 & 0 & u_{j 0} & 0 & \frac{\delta_{2 j}}{\rho_{0}} \\
0 & 0 & 0 & u_{j 0} & \frac{\delta_{3 j}}{\rho_{0}} \\
0 & \gamma \delta_{1 j} p_{0} & \gamma \delta_{2 j} p_{0} & \gamma \delta_{3 j} p_{0} & u_{j 0}
\end{array}\right),  \tag{2.76}\\
& A_{j}^{\prime}=\left[\begin{array}{ccccc}
u_{j}^{\prime} & \delta_{1 j} \rho^{\prime} & \delta_{2 j} \rho^{\prime} & \delta_{3 j} \rho^{\prime} & 0 \\
0 & u_{j}^{\prime} & 0 & 0 & -\delta_{1 j} \frac{\rho^{\prime}}{\rho_{2}^{2}} \\
0 & 0 & u_{j}^{\prime} & 0 & -\delta_{2 j} \frac{\rho}{\rho_{2}^{2}} \\
0 & 0 & 0 & u_{j}^{\prime} & -\delta_{3 j} \frac{\rho}{\rho_{0}^{2}} \\
0 & \gamma \delta_{1 j} p^{\prime} & \gamma \delta_{2 j} p^{\prime} & \gamma \delta_{3 j} p^{\prime} & u_{j}^{\prime}
\end{array}\right] . \tag{2.77}
\end{align*}
$$

The source vector $\mathbf{Q}^{\prime}$, which is only slightly different from Eq.(2.73), is given by:

$$
\mathbf{Q}^{\prime}=\left(\begin{array}{c}
S_{m}^{\prime}  \tag{2.78}\\
\frac{1}{\rho_{0}}\left(S_{1}^{\prime}-u_{0} S_{m}^{\prime}-u^{\prime} S_{m 0}\right)-\frac{\rho^{\prime}}{\rho_{9}^{2}}\left(S_{10}-u_{0} S_{m 0}\right) \\
\frac{1}{\rho_{0}}\left(S_{2}^{\prime}-v_{0} S_{m}^{\prime}-v^{\prime} S_{m 0}\right)-\frac{\rho}{\rho_{0}^{2}}\left(S_{20}-v_{0} S_{m 0}\right) \\
\frac{1}{\rho_{0}}\left(S_{3}^{\prime}-w_{0} S_{m}^{\prime}-w^{\prime} S_{m 0}\right)-\frac{\rho^{\prime}}{\rho_{0}^{2}}\left(S_{30}-w_{0} S_{m 0}\right) \\
(\gamma-1)\left\{S_{e}^{\prime}-u_{k}^{\prime} S_{k 0}-u_{k 0} S_{k}^{\prime}+\frac{1}{2} u_{k 0} u_{k 0} S_{m}^{\prime}+u_{k 0} u_{k}^{\prime} S_{m 0}\right\}
\end{array}\right) .
$$

In [31], Ewert et al. use the above equations. However the matrix $A_{j}^{\prime}$, used by Ewert et al. differs slightly from Eq.(2.77). This difference is caused by the fact that Ewert et al. remove the sources before linearizing the equations. The source term employed by Ewert et al. is merely added in the end. In [30], Ewert et al. write the above equations in yet another form; they replace the term $A_{j}^{\prime} \frac{\partial \mathbf{q}_{0}}{\partial x_{j}}$ by $C^{T} \mathbf{q}^{\prime}$, where the matrix $C$ contains spatial derivatives of the mean flow.

Most frequently in literature the LEE appear in the form of Eq.(2.75), where use is made of additional assumptions, such as, for example, the assumption of a stationary mean flow, an uniform mean flow or a quiescent mean flow ( $u_{i 0}=0$ ). We could here have included a list of references in which the LEE appear in the form of Eq.(2.75), however such a list would be very long and have no added value.

### 2.3.2 Linearizing the Euler equations in quasi-linear form for conservative variables

The Euler equations written in quasi-linear form are given by:

$$
\begin{equation*}
\frac{\partial \mathbf{U}}{\partial t}+B_{j} \frac{\partial \mathbf{U}}{\partial x_{j}}=\mathbf{S}(\mathbf{U}) \tag{2.79}
\end{equation*}
$$

where $B_{j}$ denotes the flux Jacobian:

$$
\begin{equation*}
B_{j} \equiv \frac{d \mathbf{F}_{j}}{d \mathbf{U}}(\mathbf{U}) \tag{2.80}
\end{equation*}
$$

The Jacobians and the derivation of these Jacobians are presented in appendix A.1.

## Formulation 4

Using Eq.(2.64) and Eq.(2.62) to linearize, we obtain:

$$
\begin{equation*}
\frac{\partial}{\partial t}\left(\mathbf{U}_{0}+\mathbf{U}^{\prime}\right)+B_{j}\left(\mathbf{U}_{0}+\mathbf{U}^{\prime}\right) \frac{\partial}{\partial x_{j}}\left(\mathbf{U}_{0}+\mathbf{U}^{\prime}\right)=\mathbf{S}_{0}+\mathbf{S}^{\prime} \tag{2.81}
\end{equation*}
$$

Assuming we have a solution $\mathbf{U}_{0}$ satisfying Eq.(2.56) where

$$
\begin{equation*}
\frac{\partial \mathbf{F}_{j}\left(\mathbf{U}_{0}\right)}{\partial x_{j}}=B_{j}\left(\mathbf{U}_{0}\right) \frac{\partial \mathbf{U}_{0}}{\partial x_{j}}, \tag{2.82}
\end{equation*}
$$

and upon using the Taylor series expansion of $B_{j}\left(\mathbf{U}_{0}+\mathbf{U}^{\prime}\right)$ we obtain for the perturbation $\mathbf{U}^{\prime}$ :

$$
\begin{equation*}
\frac{\partial \mathbf{U}^{\prime}}{\partial t}+B_{j 0} \frac{\partial \mathbf{U}^{\prime}}{\partial x_{j}}=\mathbf{S}^{\prime}-\left(\frac{d B_{j}}{d \mathbf{U}}\right)_{0}^{T} \mathbf{U}^{\prime} \frac{\partial \mathbf{U}_{0}}{\partial x_{j}} \tag{2.83}
\end{equation*}
$$

where

$$
\begin{equation*}
\left(\frac{d B_{j}}{d \mathbf{U}}\right)_{0}^{T}=\left(\frac{d B_{j}}{d U_{1}}, \frac{d B_{j}}{d U_{2}}, \ldots, \frac{d B_{j}}{d U_{5}}\right)_{0}^{T} \tag{2.84}
\end{equation*}
$$

and where:

$$
\begin{equation*}
B_{j 0}=B_{j}\left(\mathbf{U}_{0}\right), \quad\left(\frac{d B_{j}}{d \mathbf{U}}\right)_{0}=\frac{d B_{j}}{d \mathbf{U}}\left(\mathbf{U}_{0}\right) \tag{2.85}
\end{equation*}
$$

So, the third-order tensor $\left(\frac{d B_{j}}{d \mathbf{U}}\right)_{0}^{T}$ is written as a vector where the entries of the vector are matrices.

## Relation between formulation 1 and 4

At first sight the results given by equations (2.67) and (2.83) look different. However, Eq.(2.67) is identical to Eq.(2.83) if:

$$
\begin{equation*}
\frac{\partial B_{j}\left(\mathbf{U}_{0}\right)}{\partial x_{j}} \mathbf{U}^{\prime} \stackrel{?}{=}\left(\frac{d B_{j}}{d \mathbf{U}}\right)_{0}^{T} \mathbf{U}^{\prime} \frac{\partial \mathbf{U}_{0}}{\partial x_{j}}+\mathcal{O}\left(\epsilon^{2}\right) \tag{2.86}
\end{equation*}
$$

Note that:

$$
\begin{equation*}
\frac{\partial B_{j}\left(\mathbf{U}_{0}\right)}{\partial x_{j}}=\left(\frac{d B_{j}}{d \mathbf{U}} \frac{\partial \mathbf{U}}{\partial x_{j}}\right)_{0}=\left(\frac{d B_{j}}{d \mathbf{U}}\right)_{0} \frac{\partial \mathbf{U}_{0}}{\partial x_{j}} \tag{2.87}
\end{equation*}
$$

In index notation Eq.(2.86) becomes:

$$
\begin{equation*}
\left(\frac{\partial\left(B_{j}\right)_{k l}}{\partial U_{m}}\right)_{0} \frac{\partial\left(U_{0}\right)_{m}}{\partial x_{j}} U_{l}^{\prime} \stackrel{?}{=}\left(\frac{\partial\left(B_{j}\right)_{k l}}{\partial U_{m}}\right)_{0} U_{m}^{\prime} \frac{\partial\left(U_{0}\right)_{l}}{\partial x_{j}} . \tag{2.88}
\end{equation*}
$$

Rewriting the right hand side of Eq.(2.88) as:

$$
\begin{equation*}
\left(\frac{\partial\left(B_{j}\right)_{k l}}{\partial U_{m}}\right)_{0} \frac{\partial\left(U_{0}\right)_{m}}{\partial x_{j}} U_{l}^{\prime} \stackrel{?}{=}\left(\frac{\partial\left(B_{j}\right)_{k m}}{\partial U_{l}}\right)_{0} \frac{\partial\left(U_{0}\right)_{m}}{\partial x_{j}} U_{l}^{\prime} \tag{2.89}
\end{equation*}
$$

from which it follows that Eq.(2.67) is identical to Eq.(2.83) if:

$$
\begin{equation*}
\left(\frac{\partial\left(B_{j}\right)_{k l}}{\partial U_{m}}\right)_{0}=\left(\frac{\partial\left(B_{j}\right)_{k m}}{\partial U_{l}}\right)_{0} . \tag{2.90}
\end{equation*}
$$

From the definition of $B_{j}$, Eq.(2.80), we have:

$$
\begin{equation*}
\frac{\partial\left(B_{j}\right)_{k l}}{\partial U_{m}}=\frac{\partial^{2}\left(f_{j}\right)_{k}}{\partial U_{l} \partial U_{m}} \tag{2.91}
\end{equation*}
$$

and

$$
\begin{equation*}
\frac{\partial\left(B_{j}\right)_{k m}}{\partial U_{l}}=\frac{\partial^{2}\left(f_{j}\right)_{k}}{\partial U_{m} \partial U_{l}} . \tag{2.92}
\end{equation*}
$$

Since

$$
\begin{equation*}
\frac{\partial^{2}\left(f_{j}\right)_{k}}{\partial U_{l} \partial U_{m}}=\frac{\partial^{2}\left(f_{j}\right)_{k}}{\partial U_{m} \partial U_{l}}, \tag{2.93}
\end{equation*}
$$

it follows that Eq.(2.67) is indeed identical to Eq.(2.83).

### 2.3.3 Linearizing the Euler equations for primitive variables

In appendix A. 2 the derivation of the Euler equations in terms of primitive variables from the Euler equations in conservation form for conservative variables is presented. In index notation the Euler equations in terms of primitive variables are given by Eq.(1.23):

$$
\begin{align*}
& \frac{\partial \rho}{\partial t}+u_{j} \frac{\partial \rho}{\partial x_{j}}+\rho \frac{\partial u_{j}}{\partial x_{j}}=S_{m} \\
& \frac{\partial u_{i}}{\partial t}+u_{j} \frac{\partial u_{i}}{\partial x_{j}}+\frac{1}{\rho} \frac{\partial p}{\partial x_{i}}=\frac{1}{\rho}\left(S_{i}-u_{i} S_{m}\right), \quad i=1,2,3 \\
& \frac{\partial p}{\partial t}+u_{j} \frac{\partial p}{\partial x_{j}}+\gamma p \frac{\partial u_{j}}{\partial x_{j}}=(\gamma-1)\left\{S_{e}-u_{i} S_{i}+\frac{1}{2} u_{i} u_{i} S_{m}\right\} \tag{2.94}
\end{align*}
$$

In vector notation the equations can be written as: by:

$$
\begin{equation*}
\frac{\partial \mathbf{q}}{\partial t}+A_{j} \frac{\partial \mathbf{q}}{\partial x_{j}}=\mathbf{Q} \tag{2.95}
\end{equation*}
$$

where

$$
A_{j}=\left(\begin{array}{ccccc}
u_{j} & \delta_{1 j} \rho & \delta_{2 j} \rho & \delta_{3 j} \rho & 0  \tag{2.96}\\
0 & u_{j} & 0 & 0 & \frac{\delta_{1 j}}{\rho} \\
0 & 0 & u_{j} & 0 & \frac{\delta_{2 j}}{\rho} \\
0 & 0 & 0 & u_{j} & \frac{\delta_{3 j}}{\rho} \\
0 & \gamma \delta_{1 j} p & \gamma \delta_{2 j} p & \gamma \delta_{3 j} p & u_{j}
\end{array}\right)
$$

and

$$
\mathbf{Q}=\left(\begin{array}{c}
S_{m}  \tag{2.98}\\
\frac{1}{\rho}\left(S_{1}-u S_{m}\right) \\
\frac{1}{\rho}\left(S_{2}-v S_{m}\right) \\
\frac{1}{\rho}\left(S_{3}-w S_{m}\right) \\
(\gamma-1)\left\{S_{e}-u_{k} S_{k}+\frac{1}{2}|\vec{u}|^{2} S_{m}\right\}
\end{array}\right) .
$$

## Formulation 5

In order to linearize Eq.(2.95) we assume we have a solution $\mathbf{q}_{0}$ satisfying:

$$
\begin{equation*}
\frac{\partial \mathbf{q}_{0}}{\partial t}+A_{j 0} \frac{\partial \mathbf{q}_{0}}{\partial x_{j}}=\mathbf{Q}_{0} \tag{2.99}
\end{equation*}
$$

and substitute $\mathbf{q}=\mathbf{q}_{0}+\mathbf{q}^{\prime}$ and $\mathbf{Q}=\mathbf{Q}_{0}+\mathbf{Q}^{\prime}$ into Eq.(2.95) and use the relation:

$$
\begin{equation*}
A_{j}\left(\mathbf{q}_{0}+\mathbf{q}^{\prime}\right)=A_{j}\left(\mathbf{q}_{0}\right)+\left(\frac{d A_{j}}{d \mathbf{q}}\right)_{0}^{T} \mathbf{q}^{\prime}+\mathcal{O}\left(\varepsilon^{2}\right) \tag{2.100}
\end{equation*}
$$

to obtain for the perturbation:

$$
\begin{equation*}
\frac{\partial \mathbf{q}^{\prime}}{\partial t}+A_{j 0} \frac{\partial \mathbf{q}^{\prime}}{\partial x_{j}}+A_{j}^{\prime} \frac{\partial \mathbf{q}_{0}}{\partial x_{j}}=\mathbf{Q}^{\prime} \tag{2.101}
\end{equation*}
$$

It can be shown that:

$$
\begin{equation*}
A_{j}^{\prime}=\left(\frac{d A_{j}}{d \mathbf{q}}\right)_{0}^{T} \mathbf{q}^{\prime} \tag{2.102}
\end{equation*}
$$

In the above equations, Eqs.(2.100) and (2.102), we have written the third-order tensor $\left(\frac{d A_{j}}{d \mathbf{q}}\right)_{0}^{T}$ as a vector with matrices as entries:

$$
\begin{equation*}
\left(\frac{d A_{j}}{d \mathbf{q}}\right)_{0}^{T}=\left(\frac{d A_{j}}{d q_{1}}, \frac{d A_{j}}{d q_{2}}, \ldots, \frac{d A_{j}}{d q_{5}}\right)_{0}^{T} \tag{2.103}
\end{equation*}
$$

## Relation between formulation 3 and 5

Comparing Eq.(2.75) with Eq.(2.101) it is immediately clear that formulations 3 and 5 are equivalent.

## Relation between formulation 4 and 5

The relation between formulations 4 and 5 is most conveniently shown starting from the Euler equations in quasi-linear form, Eq.(2.79):

$$
\frac{\partial \mathbf{U}}{\partial t}+B_{j} \frac{\partial \mathbf{U}}{\partial x_{j}}=\mathbf{S} .
$$

These equations can also be written as:

$$
\begin{equation*}
\frac{d \mathbf{U}}{d \mathbf{q}} \frac{\partial \mathbf{q}}{\partial t}+B_{j} \frac{d \mathbf{U}}{d \mathbf{q}} \frac{\partial \mathbf{q}}{\partial x_{j}}=\mathbf{S} \tag{2.104}
\end{equation*}
$$

where we define.

$$
\mathcal{Z}^{-1} \equiv \frac{d \mathbf{U}}{d \mathbf{q}}=\left[\begin{array}{ccccc}
1 & 0 & 0 & 0 & 0  \tag{2.105}\\
u & \rho & 0 & 0 & 0 \\
v & 0 & \rho & 0 & 0 \\
w & 0 & 0 & \rho & 0 \\
\frac{1}{2}\left|\vec{u}^{2}\right| & \rho u & \rho v & \rho w & \frac{1}{\gamma-1}
\end{array}\right]
$$

Multiplication of Eq.(2.104) by $\mathcal{Z}$ then results in

$$
\begin{equation*}
\frac{\partial \mathbf{q}}{\partial t}+A_{j} \frac{\partial \mathbf{q}}{\partial x_{j}}=\mathcal{Z} \mathbf{S} \tag{2.106}
\end{equation*}
$$

where it can be shown that

$$
\begin{equation*}
A_{j}=\mathcal{Z} B_{j} \mathcal{Z}^{-1} \tag{2.107}
\end{equation*}
$$

with

$$
\mathcal{Z}=\left[\begin{array}{ccccc}
1 & 0 & 0 & 0 & 0  \tag{2.108}\\
-\frac{u}{\rho} & \frac{1}{\rho} & 0 & 0 & 0 \\
-\frac{v}{\rho} & 0 & \frac{1}{\rho} & 0 & 0 \\
-\frac{w}{\rho} & 0 & 0 & \frac{1}{\rho} & 0 \\
\frac{\gamma-1}{2}\left|\vec{u}^{2}\right| & -(\gamma-1) u & -(\gamma-1) v & -(\gamma-1) w & \gamma-1
\end{array}\right] .
$$

In order to linearize Eq.(2.75) we assume we have a solution $\mathbf{q}_{0}$ satisfying:

$$
\begin{equation*}
\frac{\partial \mathbf{q}_{0}}{\partial t}+A_{j 0} \frac{\partial \mathbf{q}_{0}}{\partial x_{j}}=\mathcal{Z}_{0} \mathbf{S}_{0} \tag{2.109}
\end{equation*}
$$

and substitute $\mathbf{q}=\mathbf{q}_{0}+\mathbf{q}^{\prime}$ and $\mathbf{S}=\mathbf{S}_{0}+\mathbf{S}^{\prime}$ into Eq.(2.106) and use Eq.(2.100) and the relation:

$$
\begin{equation*}
\mathcal{Z}\left(\mathbf{q}_{0}+\mathbf{q}^{\prime}\right)=\mathcal{Z}\left(\mathbf{q}_{0}\right)+\left(\frac{d \mathcal{Z}}{d \mathbf{q}}\right)_{0}^{T} \mathbf{q}^{\prime}+\mathcal{O}\left(\varepsilon^{2}\right) \tag{2.110}
\end{equation*}
$$

to obtain for the perturbation:

$$
\begin{equation*}
\frac{\partial \mathbf{q}^{\prime}}{\partial t}+A_{j 0} \frac{\partial \mathbf{q}^{\prime}}{\partial x_{j}}+A_{j}^{\prime} \frac{\partial \mathbf{q}_{0}}{\partial x_{j}}=\mathcal{Z}_{0} \mathbf{S}^{\prime}+\mathcal{Z}^{\prime} \mathbf{S}_{0} \tag{2.111}
\end{equation*}
$$

We have $A_{j}^{\prime}$ given by Eq.(2.102), furthermore it can be shown that

$$
\begin{equation*}
\mathcal{Z}^{\prime}=\left(\frac{d \mathcal{Z}}{d \mathbf{q}}\right)_{0}^{T} \mathbf{q}^{\prime} \tag{2.112}
\end{equation*}
$$

It is easily shown that $\mathbf{Q}^{\prime}$, which is presented in Eq.(2.78), satisfies the relation $\mathbf{Q}^{\prime}=\mathcal{Z}_{0} \mathbf{S}^{\prime}+$ $\mathcal{Z}^{\prime} \mathbf{S}_{0}$.

### 2.4 Linearized Euler equations and sources of sound

In the preceding section we have seen that a linear set of equations can be derived from the Euler equations in various different ways. We have shown that, by rewriting one of these linear sets of equations, one of the other formulations can be obtained (to order $\mathcal{O}\left(\varepsilon^{2}\right)$ ), without any additional assumptions and/or approximations. All presented formulations describe the same flow and any of the formulations may be labeled the linearized Euler equations.

So far we have only discussed the propagation of sound, which is our main interest. But what about the sources of sound? And which terms in the LEE can be identified to represent an aeroacoustic source? In preceding sections we introduced sources for the continuity, momentum and energy equations. The sources for the Euler and/or linearized Euler equations, however, are not always sources of sound as well (but they may influence propagation). At the end of the nineteenth century it was generally believed that mechanical vibrations of structures were the main sources of sound. It was not until the famous articles of Sir James Lighthill, titled "On sound generated aerodynamically. Parts I \& II" ([54] and [55]), that unsteady flow was identified as, so-called, aeroacoustic source of sound. Lighthill derived a wave equation, which is formally exact, from the conservation laws of mass and momentum to show the influence of unsteady flow. Before we will identify the sources of sound in the LEE, by recasting the LEE into a wave equation, we will briefly present the derivation of Lighthill's equation.

The objective in Lighthill's first paper (Part I; [54]) was to estimate the radiated sound from a given fluctuating fluid flow. Hereto Lighthill considered a limited volume occupied by a fluctuating fluid embedded in a very large volume of fluid at rest. Assuming that a listener, who detects the acoustic field, is surrounded by the uniform stagnant fluid, the acoustic field at the listener's location is accurately described by the homogeneous wave equation. Subsequently Lighthill derived a non-homogeneous wave equation from the exact equations of motionHence, it is assumed that the uniform stagnant fluid ( $u_{i \infty}=0, i=1,2,3$ ) with speed of sound $c_{\infty}$, density $\rho_{\infty}$ and pressure $p_{\infty}$, at the listener's location, extends into the entire volume, and that any departure from this, assumed ideal, acoustic behavior described by the homogeneous wave equation is equivalent to a source of sound for the observer. The non-homogeneous wave equation is obtained from the Navier-Stokes equations by taking the time-derivative of Eq.(2.1) and upon subtracting from the result the divergence of Eq.(2.2) to obtain:

$$
\begin{equation*}
\frac{\partial^{2} \rho}{\partial t^{2}}=\frac{\partial^{2}}{\partial x_{i} \partial x_{j}}\left(\rho u_{i} u_{j}+p \delta_{i j}-\tau_{i j}\right)+\frac{\partial S_{m}}{\partial t}-\frac{\partial S_{i}}{\partial x_{i}} . \tag{2.113}
\end{equation*}
$$

Subtracting from both sides a term:

$$
\begin{equation*}
c_{\infty}^{2} \frac{\partial^{2} \rho}{\partial x_{i} \partial x_{i}} \tag{2.114}
\end{equation*}
$$

it is obtained:

$$
\begin{equation*}
\frac{\partial^{2} \rho}{\partial t^{2}}-c_{\infty}^{2} \frac{\partial^{2} \rho}{\partial x_{i} \partial x_{i}}=\frac{\partial^{2}}{\partial x_{i} \partial x_{j}}\left(\rho u_{i} u_{j}+\left(p-c_{\infty}^{2} \rho\right) \delta_{i j}-\tau_{i j}\right)+\frac{\partial S_{m}}{\partial t}-\frac{\partial S_{i}}{\partial x_{i}} \tag{2.115}
\end{equation*}
$$

where we used ( $c_{\infty}=$ constant $)$ :

$$
\begin{equation*}
c_{\infty}^{2} \frac{\partial^{2} \rho}{\partial x_{i} \partial x_{i}}=c_{\infty}^{2} \frac{\partial^{2} \rho}{\partial x_{i} \partial x_{i}}=\frac{\partial^{2}}{\partial x_{i} \partial x_{j}}\left(c_{\infty}^{2} \rho \delta_{i j}\right) . \tag{2.116}
\end{equation*}
$$

The non-homogeneous wave equation for the density perturbation can be written as:

$$
\begin{equation*}
\frac{\partial^{2} \rho}{\partial t^{2}}-c_{\infty}^{2} \frac{\partial^{2} \rho}{\partial x_{i} \partial x_{i}}=\frac{\partial^{2} T_{i j}}{\partial x_{i} \partial x_{j}}+\frac{\partial S_{m}}{\partial t}-\frac{\partial S_{i}}{\partial x_{i}} \tag{2.117}
\end{equation*}
$$

where Lighthill's stress tensor is defined by:

$$
\begin{equation*}
T_{i j}=\rho u_{i} u_{j}-\tau_{i j}+\left(p-c_{\infty}^{2} \rho\right) \delta_{i j} \tag{2.118}
\end{equation*}
$$

In the original derivation, Lighthill did not take the influence of an external mass source ( $S_{m}$ ) and of external forces $\left(S_{i}\right)$ into account. As explained in section 2.2 , in general $S_{m}=0$, since we consider a non-relativistic situation in which mass is conserved. However, the mass source term $S_{m}$ can be used to represent a complex process ([70]) such as, for example, the action of a pulsating sphere or of mass injection. The source term $S_{i}, i=1,2,3$ describes an external force field, which may in general be assumed absent. From Eq.(2.117) it is observed that an steady mass source does not produce sound. In addition, when the external force field $S_{i}, i=1,2,3$ is given by, in vector notation, $\mathbf{S}_{f}=\left(S_{1}, S_{2}, S_{3}\right)^{T}=\boldsymbol{\nabla} \times \mathbf{F}$, the force field produces no sound because $\boldsymbol{\nabla} \cdot(\boldsymbol{\nabla} \times \mathbf{F})=0$.

In Eq.(2.117) we distinguish, apart from the external mass source and external force field, three aeroacoustic processes which result in sources of sound:
i) the non-linear convective forces described by the fluctuating Reynolds stress tensor $\rho u_{i} u_{j}$,
ii) the viscous stresses $\tau_{i j}$,
iii) and the deviation from an isentropic behavior $p-c_{\infty}^{2} \rho$.

It should be noted ([54]) that all effects such as the convection of sound by turbulent flow, or the variations of the speed of sound within it, are taken into account in Eq.(2.117), because these effects are incorporated as equivalent applied stresses. This is evidently true ([54]), since the equations are exact for any arbitrary fluid motion. However, for an airflow embedded in a uniform atmosphere at rest, the Lighthill stress term $T_{i j}$ (Eq.(2.118)) can be neglected outside the flow itself!

To identify sources of sound present in the LEE, the LEE can also be cast into a wave equation. For the background flow, we introduce the following 'time averaged" flow:

$$
\begin{equation*}
q_{0}(\mathbf{x}, t) \equiv \int_{t-\Delta t}^{t+\Delta t} q(\mathbf{x}, \tau) d \tau \tag{2.119}
\end{equation*}
$$

where $q$ is either the density, the velocity component in $x, y$ or $z$-direction or the pressure. The time increment $\Delta t$ is chosen such that:

$$
\begin{equation*}
\frac{\lambda}{c_{\infty}} \ll \Delta t \ll \frac{L}{U} \tag{2.120}
\end{equation*}
$$

where $\lambda$ is the shortest acoustic wave length, $L$ and $U$ are a characteristic length scale and the velocity of the considered object, respectively. For example, when considering the noise
propagating away from a passing airplane, $L$ is related to the size of the airplane and $U$ is the velocity with which the airplane travels.

It is assumed that there are no mean flow or background flow sources ( $\mathbf{S}_{0}=\mathbf{0}$ ). In addition, at large distance away from the fluctuating fluid we assume:

$$
\begin{equation*}
|\mathbf{x}| \rightarrow \infty \quad\left(\rho_{0}, u_{i 0}, p_{0}\right)^{T} \rightarrow\left(\rho_{\infty}, 0, p_{\infty}\right)^{T} \tag{2.121}
\end{equation*}
$$

To obtain a wave equation from the LEE consider the linear continuity and momentum equations as presented in Eqs.(2.59), written in the following form:

$$
\begin{align*}
& \frac{\partial \rho^{\prime}}{\partial t}+\frac{\partial}{\partial x_{j}}\left(\rho u_{j}\right)^{\prime}=S_{m}^{\prime},  \tag{2.122}\\
& \frac{\partial}{\partial t}\left(\rho u_{i}\right)^{\prime}+\frac{\partial}{\partial x_{j}}\left(\left(\rho u_{i} u_{j}\right)^{\prime}+p^{\prime} \delta_{i j}\right)=S_{i}^{\prime} \tag{2.123}
\end{align*}
$$

where we used the convenient short hand notation:

$$
\begin{align*}
& \left(\rho u_{i}\right)^{\prime} \equiv \rho_{0} u_{i}^{\prime}+u_{i 0} \rho^{\prime}  \tag{2.124}\\
& \left(\rho u_{i} u_{j}\right)^{\prime} \equiv \rho_{0} u_{i 0} u_{j}^{\prime}+\rho_{0} u_{i}^{\prime} u_{j 0}+\rho^{\prime} u_{i 0} u_{j 0} . \tag{2.125}
\end{align*}
$$

Upon taking the time-derivative of Eq.(2.122) and subtracting from it the divergence of Eq.(2.123) we obtain:

$$
\begin{equation*}
\frac{\partial^{2} \rho^{\prime}}{\partial t^{2}}-\frac{\partial^{2}}{\partial x_{i} \partial x_{j}}\left(\left(\rho u_{i} u_{j}\right)^{\prime}+p^{\prime} \delta_{i j}\right)=\frac{\partial S_{m}^{\prime}}{\partial t}-\frac{\partial S_{i}^{\prime}}{\partial x_{i}} \tag{2.126}
\end{equation*}
$$

Let us now define a constant speed of sound $c_{\infty}$ at the listener's location. We can subtract from both sides of Eq.(2.126) a term:

$$
\begin{equation*}
c_{\infty}^{2} \frac{\partial^{2} \rho^{\prime}}{\partial x_{i} \partial x_{i}}, \tag{2.127}
\end{equation*}
$$

to obtain:

$$
\begin{equation*}
\frac{\partial^{2} \rho^{\prime}}{\partial t^{2}}-c_{\infty}^{2} \frac{\partial^{2} \rho^{\prime}}{\partial x_{i} \partial x_{i}}=\frac{\partial^{2}}{\partial x_{i} \partial x_{j}}\left(\left(\rho u_{i} u_{j}\right)^{\prime}+\left(p^{\prime}-c_{\infty}^{2} \rho^{\prime}\right) \delta_{i j}\right)+\frac{\partial S_{m}^{\prime}}{\partial t}-\frac{\partial S_{i}^{\prime}}{\partial x_{i}} \tag{2.128}
\end{equation*}
$$

where we used the relation of Eq.(2.116). Eq.(2.128) shows great resemblance with the famous Lighthill's equation, Eq.(2.117). However, the sources in the Lighthill's equation do not have to be small. In addition in Eq.(2.128) the viscous stresses are not present. In section 2.2.1 we have shown, however, that the influence of viscous stresses is, in most cases, negligible, as is also concluded by Lighthill ([54]).

From Eq.(2.121) it is observed that at large distance away from the fluctuating fluid the fluctuating Reynolds stress tensor given by the linear wave equation disappears:

$$
\begin{equation*}
|\mathbf{x}| \rightarrow \infty \quad\left(\rho u_{i} u_{j}\right)^{\prime}=\rho_{0} u_{i 0} u_{j}^{\prime}+\rho_{0} u_{i}^{\prime} u_{j 0}+\rho^{\prime} u_{i 0} u_{j 0} \quad \rightarrow 0 \tag{2.129}
\end{equation*}
$$

When interest is in the propagation of sound in a sheared mean flow, it should be noted that the term $\rho_{0} u_{i}^{\prime} u_{j}^{\prime}$, which we have neglected, might not be small anymore. In such a case
$\varepsilon$ is apparently not small enough to neglect terms of order $\mathcal{O}\left(\varepsilon^{2}\right)$. This explains the choice of sources in by Bailly et al. ([10]). In section 2.3.1 the sources used in reference [10] have been briefly discussed.

The fluctuating Reynolds stress tensor furthermore plays an important role in turbulent flows. More about the fluctuating Reynolds stress tensor can be found in Lighthill's part II ([55]), which deals with turbulence as source of sound.

In [70] the source term $S_{m}^{\prime}$ has been written out in some detail when it describes the process of mass injection.

### 2.5 Simplifi ed linearized Euler equations in dimensionless form

The LEE in primitive variables and in vector notation are given by Eq.(2.75) (and equivalently by Eq.(2.101) and Eq.(2.111)):

$$
\begin{equation*}
\frac{\partial \mathbf{q}^{\prime}}{\partial t}+A_{j 0} \frac{\partial \mathbf{q}^{\prime}}{\partial x_{j}}+A_{j}^{\prime} \frac{\partial \mathbf{q}_{0}}{\partial x_{j}}=Q^{\prime} \tag{2.130}
\end{equation*}
$$

where $Q^{\prime}$ is given by Eq.(2.78). Furthermore, $Q^{\prime}$ satisfies the relation:

$$
\begin{equation*}
Q^{\prime}=Z_{0} \mathbf{S}^{\prime}+Z^{\prime} \mathbf{S}_{0}, \tag{2.131}
\end{equation*}
$$

as explained when describing the relation between formulation 4 and 5 in section 2.3.3. The coefficients of the matrices $A_{j}^{\prime}$ and $Z^{\prime}$ in Eq.(2.130) are functions of the primitive perturbation variables $\mathbf{q}^{\prime}$, i.e. $A_{j}^{\prime}\left(\mathbf{q}^{\prime}\right)$ and $Z^{\prime}\left(\mathbf{q}^{\prime}\right)$. For the numerical treatment of the LEE it is convenient to write $A_{j 0} \frac{\partial \mathbf{q}^{\prime}}{\partial x_{j}}$ and $Z^{\prime} \mathbf{S}_{0}$ both in terms of a product of a matrix, which is independent of the perturbation variables, and the primitive perturbation vector $\mathbf{q}^{\prime}$. This can be done as follows: It can be shown, employing Eq.(2.108), that for $Z^{\prime} \mathbf{S}_{0}$ we can write:

$$
\begin{equation*}
Z^{\prime} \mathbf{S}_{0} \equiv Y_{0} \mathbf{q}^{\prime} \tag{2.132}
\end{equation*}
$$

where
$Y_{0}=\left[\begin{array}{ccccc}0 & 0 & 0 & 0 & 0 \\ \frac{u_{0} S_{m 0}-S_{10}}{\rho_{0}^{2}} & \frac{-S_{m 0}}{\rho_{0}} & 0 & 0 & 0 \\ \frac{v_{0} S_{m 0}-S_{20}}{\rho_{0}^{2}} & 0 & \frac{-S_{m 0}}{\rho_{0}} & 0 & 0 \\ \frac{w_{0} S_{m 0}}{\rho_{0}^{2}} & 0 & 0 & \frac{-S_{30}}{\rho_{0}} & 0 \\ 0 & (\gamma-1)\left(u_{0} S_{m 0}-S_{10}\right) & (\gamma-1)\left(v_{0} S_{m 0}-S_{20}\right) & (\gamma-1)\left(w_{0} S_{m 0}-S_{30}\right) & 0\end{array}\right]$.
When there are no sources in the mean flow $\mathbf{S}_{0}=\mathbf{0}$ and $Y_{0}=0$, i.e. all elements of the matrix $Y_{0}$ are zero.

Employing Eq.(2.102), the last term at the left-hand side of Eq.(2.130) can be written as:

$$
\begin{equation*}
A_{j}^{\prime} \frac{\partial \mathbf{q}_{0}}{\partial x_{j}}=\left(\frac{d A_{j}}{d \mathbf{q}}\right)_{0}^{T} \mathbf{q}^{\prime} \frac{\partial \mathbf{q}_{0}}{\partial x_{j}} \tag{2.134}
\end{equation*}
$$

where we have written the third-order tensor $\left(\frac{d A_{j}}{d \mathbf{q}}\right)_{0}^{T}$ as a vector with matrices as entries, Eq.(2.103):

$$
\left(\frac{d A_{j}}{d \mathbf{q}}\right)_{0}^{T}=\left(\frac{d A_{j}}{d q_{1}}, \frac{d A_{j}}{d q_{2}}, \ldots, \frac{d A_{j}}{d q_{5}}\right)_{0}^{T}
$$

It can be shown that Eq.(2.134) can be written as:

$$
\begin{equation*}
\left(\frac{d A_{j}}{d \mathbf{q}}\right)_{0}^{T} \mathbf{q}^{\prime} \frac{\partial \mathbf{q}_{0}}{\partial x_{j}}=C_{0} \mathbf{q}^{\prime} \tag{2.135}
\end{equation*}
$$

where

$$
C_{0}=\left[\begin{array}{ccccc}
\frac{\partial u_{j 0}}{\partial x_{j}} & \frac{\partial \rho_{0}}{\partial x} & \frac{\partial \rho_{0}}{\partial y} & \frac{\partial \rho_{0}}{\partial z} & 0  \tag{2.136}\\
\frac{-1}{\rho_{0}^{2}} \frac{\partial p_{0}}{\partial x} & \frac{\partial u_{0}}{\partial x} & \frac{\partial u_{0}}{\partial y} & \frac{\partial u_{0}}{\partial z} & 0 \\
\frac{-1}{\rho_{0}^{2}} \frac{\partial p_{0}}{\partial y} & \frac{\partial v_{0}}{\partial x} & \frac{\partial v_{0}}{\partial y} & \frac{\partial v_{0}}{\partial z} & 0 \\
\frac{-1}{\rho_{0}^{2}} \frac{\partial p_{0}}{\partial z} & \frac{\partial w_{0}}{\partial x} & \frac{\partial w_{0}}{\partial y} & \frac{\partial w_{0}}{\partial z} & 0 \\
0 & \frac{\partial p_{0}}{\partial x} & \frac{\partial p_{0}}{\partial y} & \frac{\partial p_{0}}{\partial z} & \gamma \frac{\partial u_{j 0}}{\partial x_{j}}
\end{array}\right] .
$$

The LEE can now be written in the form:

$$
\begin{equation*}
\frac{\partial \mathbf{q}^{\prime}}{\partial t}+A_{j 0} \frac{\partial \mathbf{q}^{\prime}}{\partial x_{j}}+C_{0} \mathbf{q}^{\prime}-Y_{0} \mathbf{q}^{\prime}=Z_{0} \mathbf{S}^{\prime} \tag{2.137}
\end{equation*}
$$

Ewert et al. [30] use the LEE in the form of Eq.(2.137). Finally Eq.(2.137) can be written in the form:

$$
\begin{equation*}
\frac{\partial \mathbf{q}^{\prime}}{\partial t}+\frac{\partial}{\partial x_{j}}\left(A_{j 0} \mathbf{q}^{\prime}\right)+D_{0} \mathbf{q}^{\prime}-Y_{0} \mathbf{q}^{\prime}=Z_{0} \mathbf{S}^{\prime} \tag{2.138}
\end{equation*}
$$

where

$$
\begin{equation*}
D_{0} \equiv C_{0}-\frac{\partial A_{j 0}}{\partial x_{j}} \tag{2.139}
\end{equation*}
$$

and

$$
D_{0}=\left[\begin{array}{ccccc}
0 & 0 & 0 & 0 & 0  \tag{2.140}\\
\frac{-1}{\rho_{0}^{2}} \frac{\partial p_{0}}{\partial x} & -\left(\frac{\partial v_{0}}{\partial y}+\frac{\partial w_{0}}{\partial z}\right) & \frac{\partial u_{0}}{\partial y} & \frac{\partial u_{0}}{\partial z} & \frac{1}{\rho_{0}^{2}} \frac{\partial p_{0}}{\partial x} \\
\frac{-1}{\rho_{0}^{2}} \frac{\partial p_{0}}{\partial y} & \frac{\partial v_{0}}{\partial x} & -\left(\frac{\partial u_{0}}{\partial x}+\frac{\partial w_{0}}{\partial z}\right) & \frac{\partial v_{0}}{\partial z} & \frac{1}{\rho_{0}^{2}} \frac{\partial p_{0}}{\partial x} \\
\frac{-1}{\rho_{0}^{2}} \frac{\partial p_{0}}{\partial z} & \frac{\partial w_{0}}{\partial x} & \frac{\partial w_{0}}{\partial y} & -\left(\frac{\partial u_{0}}{\partial x}+\frac{\partial v_{0}}{\partial y}\right) & \frac{1}{\rho_{0}^{2}} \frac{\partial p_{0}}{\partial x} \\
0 & (1-\gamma) \frac{\partial p_{0}}{\partial x} & (1-\gamma) \frac{\partial p_{0}}{\partial y} & (1-\gamma) \frac{\partial p_{0}}{\partial z} & (\gamma-1) \frac{\partial u_{j 0}}{\partial x_{j}}
\end{array}\right] .
$$

The coefficients of the matrix $D_{0}$ are functions of the gradient of the mean flow, i.e. $D_{0}\left(\frac{\partial \mathbf{q}_{0}}{\partial x}, \frac{\partial \mathbf{q}_{0}}{\partial y}, \frac{\partial \mathbf{q}_{0}}{\partial z}\right)$. Obviously:

$$
\begin{equation*}
\frac{\partial \mathbf{q}_{0}}{\partial x_{j}}=0 \quad \forall j \quad \Rightarrow \quad D_{0}=0 \tag{2.141}
\end{equation*}
$$

Because all elements of $D_{0}$ are linear in the components $\frac{\partial q_{m 0}}{\partial x_{j}}$, we have in addition:

$$
\begin{equation*}
\max _{j, m}\left\|\frac{\partial q_{m 0}}{\partial x_{j}}\right\| \frac{L_{r e f, \mathbf{q}_{0}}}{\left|q_{m 0}\right|}=\delta \quad \Rightarrow \quad\left\|D_{0} \mathbf{x}\right\|=\mathcal{O}(\delta) \tag{2.142}
\end{equation*}
$$

where $L_{r e f, \mathbf{q}_{0}}$ is a characteristic length scale of the background flow, $q_{m 0}$ is the $m^{t h}$-component of the primitive background flow vector $\left(\rho_{0}, u_{i 0}, p_{0}\right)^{T}$ and $\delta$ is a measure for the variations in the mean flow. Then

$$
\begin{equation*}
\left\|D_{0} \mathbf{q}^{\prime}\right\|=\mathcal{O}(\delta) \mathcal{O}(\varepsilon) \tag{2.143}
\end{equation*}
$$

In our approach we will neglect terms of order $\delta$. The error in the LEE formulation, which we introduce, tends to zero if the variations in the mean flow tend to zero, i.e. if the mean flow becomes uniform.

In both the verification problems which are considered in this thesis, the mean flow is uniform and has no sources. Under these conditions Eq.(2.138) simplifies to:

$$
\begin{equation*}
\frac{\partial \mathbf{q}^{\prime}}{\partial t}+\frac{\partial}{\partial x_{j}}\left(A_{j 0} \mathbf{q}^{\prime}\right)=Z_{0} \mathbf{S}^{\prime} \tag{2.144}
\end{equation*}
$$

where the coefficients of the matrices $A_{j 0}$ and $Z_{0}$ are constants.
In subsequent chapters we will apply the linearized Euler equations in dimensionless form. In section 2.2 it has been shown how the Navier-Stokes equations can be written in dimensionless form. In the remainder of this work it is assumed that the equations are scaled with a different set of scaling parameters to obtain dimensionless equations than in section 2.2, namely:

$$
\begin{equation*}
\rho_{r e f}, c_{r e f}, L, T_{r e f} \tag{2.145}
\end{equation*}
$$

where $\rho_{\text {ref }}, c_{r e f}$ and $L$ are an appropriate reference density, velocity and length scale, respectively, and $T_{r e f}$ is an appropriate temperature for the problem at hand.

With the above scaling parameters, densities are scaled with $\rho_{r e f}$, lengths with $L$, time with $L / c_{r e f}$ and pressure with $\rho_{r e f} c_{r e f}^{2}$. For the primitive perturbation vector we again write: $\mathbf{q}^{\prime}=\left(\rho^{\prime}, u^{\prime}, v^{\prime}, w^{\prime}, p^{\prime}\right)^{T}$. For $A_{j 0}$ we may now write:

$$
\mathbf{A}_{j 0}=\left[\begin{array}{ccccc}
M_{j} & \delta_{j 1} & \delta_{j 2} & \delta_{j 3} & 0  \tag{2.146}\\
0 & M_{j} & 0 & 0 & \delta_{1 j} \\
0 & 0 & M_{j} & 0 & \delta_{2 j} \\
0 & 0 & 0 & M_{j} & \delta_{3 j} \\
0 & \delta_{j 1} & \delta_{j 2} & \delta_{j 3} & M_{j}
\end{array}\right]
$$

where $M_{1}, M_{2}, M_{3}$ are the components of the mean flow Mach-number in x, y and z -direction, respectively, and $\delta_{i j}$ denotes the Kronecker-delta symbol.

## DISCONTINUOUS Galerkin formulation

### 3.1 Introduction

In the preceding chapter we have presented the linearized Euler Equations (LEE), which describe the propagation of sound through inhomogeneous moving fluids. The present chapter describes the numerical method developed to solve the LEE on tetrahedral elements. The method presented here is primarily based on the method developed by Atkins, Shu and Lockard, [4], [5], [6] and [7], who apply the discontinuous Galerkin (DG) finite-element method for the spatial discretization and the Runge-Kutta method for the time discretization.

As for all finite-element methods, the starting point for the DG method is a weak formulation of the problem on each of the elements in the partitioning of the domain, on which the equations are to be solved. In the DG finite-element approach the solution in each element is locally expanded in a polynomial series and equations for the polynomial coefficients are obtained (van der Vegt \& van der Ven [81]). The DG finite element method therefore not only solves equations for the flow field, but also for the moments of the flow field (van der Vegt \& van der Ven [81]). The polynomial functions, called basis functions, may be discontinuous over inter-element boundaries. For the test functions in the weak formulation the same discontinuous basis functions are used. The use of discontinuous test or basis functions has given the method its name: Discontinuous Galerkin method.

When applying the Discontinuous Gelerkin method integrals which appear in the weak formulation have to be evaluated. DG methods can be subdivided into a class in which these integrals are evaluated employing, so-called, quadrature rules and a class in which these integrals are evaluated quadrature-free. In this thesis the quadrature-free approach, as proposed by Atkins \& Shu ([5], [6], [7]), is followed. The quadrature-free approach results in a significant reduction of the computational work in comparison to the DG methods employing quadrature rules (Atkins \& Shu [6]). In addition, Atkins \& Shu ([6]) mention that not for all multidimensional elements optimal quadrature rules are readily available.

Some of the key features of the DG method are:

- Extremely compact method:

The solution within an element is not reconstructed by looking to neighboring elements. At element interfaces the information about the flow state can be directly obtained from the polynomial expansions in each element, no interpolation is therefore necessary (van der Vegt \& van der Ven [81]). The size of the stencil is independent of the desired order of accuracy. Adjacent elements only communicate with one another
through the approximate Riemann flux, which is introduced to provide coupling.

- Applicable to both structured and unstructured meshes:

The method does not require the mesh to be uniform or smooth and is therefore applicable to both structured and unstructured meshes. The method is very well suited for handling complex geometries.

- High order accuracy achievable:

Higher order accuracy, higher than the usual first or second order of most CFD-methods, can be obtained by suitably choosing the degree of the basis functions (Cockburn et al. [19]). The current trend in CFD-methods is to obtain high(er) accuracy by using reconstruction techniques or large stencils (consider for example the ENO and DRP schemes briefly described in the general introduction), which are strongly dependent on the quality of the underlying mesh (Baumann \& Oden [11]). The accuracy does not depend on the smoothness of the mesh. The accuracy in an element only depends on the size and shape of the element and on the degree to which the solution is approximated (Atkins \& Shu [6]).

- Treatment of boundary conditions is simple:

Because of the compactness of the method, elements adjacent the domain boundary can be treated in exactly the same manner as elements in the interior of the computational domain. The boundary conditions can be imposed via the approximate Riemann flux by providing the solution at the exterior side of the boundary. Often, the exterior solution is not known, in which case the boundary normal flux subjected to the boundary conditions can be reformulated in terms of interior data.

- Well suited for adaptive error control:

Because of its compactness and because inter-element continuity is not required, the DG method is a good candidate for local grid refinement (h-refinement) and/or local polynomial-degree-variation (p-refinement)(Flaherty et al. [33], van der Vegt \& van der Ven [81], Süli et al. [73]). Adaptivity (hp-refinement) is particularly interesting in hyperbolic problems, given the complexity of the structure of (possible) flow discontinuities (Cockburn et al. [19]). Where the solution is smooth p-refinement can be applied, and where the solution is non-smooth h-refinement.

## - Highly parallizable:

Since the elements are discontinuous, the "mass matrix" (of the entire system) is block diagonal. The size of the blocks is equal to the highest polynomial degree inside the corresponding element.

- Number of unknowns can be extensive:

The number of unknowns increases with increasing highest polynomial degree and can therefore be quite extensive. However, especially when applying the quadrature-free approach, the costs are acceptable.

The Discontinuous Galerkin method has become increasingly popular in the last decade, while the first reports on the DG finite-element method date back to the early 1970s (Cock-
burn et al. [19]). $\mathrm{In}^{1} 1973$ Reed \& Hill introduced the method for solving the linear neutron transport equation (which is independent of time) on triangular meshes ([66]). The first analysis of the method, applied to the linear neutron transport equation, appeared in 1974 and was conducted by LeSaint and Raviart ([53]). LeSaint and Raviart showed that, when $p$ is the highest polynomial degree and $h$ is a characteristic length scale of the elements, the rate of convergence of the solution was $h^{p}$ for general triangles and $h^{p+1}$ for Cartesian grids. More than ten years later (1986) Johnson and Pitkärata [45] analyzed the method, applied to a scalar hyperbolic equation, and proved a rate of convergence of $h^{p+\frac{1}{2}}$ for general triangulations. These results were later, in 1991, confirmed numerically by Peterson ([62]). In the mean time Richter obtained the optimal rate of convergence of $h^{p+1}$ for some structured non-Cartesian grids [68].

In the first half of the 1990s, Lin and Zhou proved convergence of the solution for hyperbolic problems in which discontinuities in the flow field were allowed to be present, [56].Recently Cockburn et al. showed that it is possible to obtain a rate of convergence of $h^{2 p+1}$ by carefully postprocessing the approximate solution.

After being successfully applied to linear and scalar hyperbolic equations, the DG has been gradually extended to systems of nonlinear equations by various authors. Cockburn, Shu et al. extended the method, together with Runga-Kutta time discretization, to hyperbolic conservation laws ([20], [21], [23], [24]). The first extension of the DG method to the three-dimensional Euler equations was by van der Vegt [80]. Van der Vegt \& van der Ven [81] combine the discretization of the Euler equations, employing the compactness of the DG method for the spatial discretization, with local grid refinement, i.e. $h$-refinement ([84]). Because of the extreme compactness of the DG method local grid refinement is well applicable. Apart from $h$-refinement the method supports $p$-refinement (local adjustment of the degree of the polynomials) and $h p$-refinement, which is a combination of $h$ - and $p$ - refinement, see for example Baumann and Oden [12]. In [11] Baumann and Oden extend the DG method, in combination with local $h p$-refinement, to the Navier-Stokes equations.

In [82] and [83], van der Vegt \& van der Ven apply the space-time discontinuous Galerkin method for the solution of the Euler equations in time-dependent flow domains. Until recently, for the full discretization of the governing equations of a problem, two main classes of Discontinuous Galerkin methods were in use. Namely, discretizations in which either the basis functions are discontinuous in space or in time. In the first class the DG method is combined with TVD (Total Variation Diminishing) Runga-Kutta time integration. In the second class the basis functions are continuous in space (Galerkin methods) and discontinuous in time. Lowrie et al. ([57]) studied the use of discontinuous basis functions in both space and time. Van der Vegt \& van der Ven also use discontinuous basis functions in both space and time, however, in a different way than Lowrie et al., which allows extension to local mesh refinement.

In this chapter the quadrature-free Discontinuous Galerkin method is applied for the spatial discretization of the linearized Euler equations (LEE). In section 2.5 of the preceding chapter,

[^2]it has been shown that the LEE can be written in the following form:
\[

$$
\begin{equation*}
\mathbf{L}(\mathbf{u}) \equiv \frac{\partial \mathbf{u}}{\partial t}+\frac{\partial \mathbf{f}_{i}(\mathbf{u})}{\partial x_{i}}=\mathbf{s}, \quad \mathbf{x} \in \Omega, t \in I_{t} \tag{3.1}
\end{equation*}
$$

\]

where

$$
\begin{equation*}
\mathbf{f}_{i}(\mathbf{u})=A_{i}\left(\mathbf{u}_{0}\right) \mathbf{u} \tag{3.2}
\end{equation*}
$$

under the assumption that the term $D_{0} \mathbf{q}^{\prime}$ in Eq.(2.138) may be neglected and that there are no sources in the mean flow. In Eq.(3.1) $\mathrm{s}\left(\in \mathbb{R}^{5}\right)$ is the source term for the LEE, $\Omega \in \mathbb{R}^{3}$ is an open domain with boundary $\partial \Omega$ and $t \in I_{t}$ denotes time, where $I_{t} \in \mathbb{R}^{+} \backslash\{0\}$. At $t=0$ initial conditions are applied. The solution vector $\mathbf{u}: \Omega \times I_{t} \longmapsto \mathbb{R}^{5}$ is given by $\mathbf{u}=\left(\rho^{\prime}, u_{1}^{\prime}, u_{2}^{\prime}, u_{3}^{\prime}, p^{\prime}\right)^{T}$, where the components of the vector denote the dimensionless aeroacoustic density perturbation, the three perturbation velocity components and the pressure perturbation, respectively. The components of vector $\mathbf{u}_{0}$ denote the dimensionless quantities related to the mean flow density, the three components of the mean flow velocity and the mean flow pressure. The perturbation velocity vector is denoted by $\boldsymbol{v}^{\prime}=\left(u_{1}^{\prime}, u_{2}^{\prime}, u_{3}^{\prime}\right)^{T}$, or, equivalently by $\boldsymbol{v}^{\prime}=\left(u^{\prime}, v^{\prime}, w^{\prime}\right)^{T}$. The velocity vector of the mean flow is denoted by $\boldsymbol{v}_{0}=\left(M_{1}, M_{2}, M_{3}\right)^{T}$, where $M_{1}, M_{2}, M_{3}$ are the components of the mean flow Machnumber in x , y and z-direction, respectively. The matrices $A_{i}: \Omega_{u} \longmapsto \Omega_{A} \subset \mathbb{R}^{5} \times \mathbb{R}^{5}$, where $\Omega_{u}=\mathbb{R}^{+} \times \mathbb{R}^{3} \times \mathbb{R}^{+}$, are defined as:

$$
A_{i}\left(\mathbf{u}_{0}\right)=\left[\begin{array}{ccccc}
M_{i} & \delta_{i 1} & \delta_{i 2} & \delta_{i 3} & 0  \tag{3.3}\\
0 & M_{i} & 0 & 0 & \delta_{1 i} \\
0 & 0 & M_{i} & 0 & \delta_{2 i} \\
0 & 0 & 0 & M_{i} & \delta_{3 i} \\
0 & \delta_{i 1} & \delta_{i 2} & \delta_{i 3} & M_{i}
\end{array}\right]
$$

where $\delta_{i j}$ denotes the Kronecker-delta symbol.
Following the introduction, section 3.2 presents the Discontinuous Galerkin space discretization of the linerized Euler equations. In the description of the method the highest polynomial degree $p$ has not been fixed. In chapter 5 and chapter 7 results are shown obtained with the numerical method in which the highest polynomial degree is $p=1$. The problems which are considered in 5 and chapter 7 both involve homogeneous background flows, so $D_{0}=0$ in Eq.(2.138). When, in later work, the mean flow quantities are nonuniform, their spatial variation must be accounted for and the term $D_{0} \mathbf{q}^{\prime}$ should be added to the left-hand-side of Eq.(3.1), as can be seen from Eq.(2.138). To ensure the formal order properties of the DG method the mean flow quantities should be represented to the same degree as the perturbation variables, i.e to degree $p$ (Atkins \& Lockard [4]). However, Atkins \& Lockard [4] argue that in wave scatter simulations, the assumption of linearity is based on the premise that the perturbation amplitude is small relative to variations in the mean flow. Therefor, according to Atkins \& Lockard [4], it is sufficient to represent the back ground flow by a lower-order polynomial in order to simulate the scatter of acoustic waves. In section 3.3 the treatment of the boundary conditions is presented and in section 3.4 the treatment of the initial conditions. Finally, in section 3.5 the multi-stage Runge-Kutta time integration algorithm is introduced. Throughout this chapter we will use the Einstein summation convention, unless stated otherwise and except for the index $j$. Furthermore we use the short notation DG for Discontinuous Galerkin (method).

### 3.2 Discontinuous Galerkin space discretization

### 3.2.1 Weak formulation of the Linearized Euler equations

Consider a domain $\Omega \subset \mathbb{R}^{3}$ in which we want to solve the LEE. Let $\mathcal{U}$ be the space of functions $\mathbf{u}(\mathbf{x}, t)$, where $\mathbf{x} \in \Omega$ and $t \in I_{t}$. We will call $\mathcal{U}$ the solution space. Let us define on $\Omega$ the space $\mathcal{V}$ which contains only spatial functions:

$$
\begin{equation*}
\mathcal{V} \equiv\left\{w(\mathbf{x}) \in L^{2}(\Omega)\right\} \tag{3.4}
\end{equation*}
$$

and define on $I_{t}$ the space $\mathcal{T}$ of only time dependent functions:

$$
\begin{equation*}
\mathcal{T} \equiv\left\{\tau(t) \in L^{2}\left(I_{t}\right)\right\} \tag{3.5}
\end{equation*}
$$

such that $\mathcal{U} \equiv \mathcal{V} \times \mathcal{T}$. We define the inner product on $\mathcal{V}$ :

$$
\begin{equation*}
(u, w)_{\Omega} \equiv \int_{\Omega} u(\mathbf{x}) w(\mathbf{x}) d \Omega, \quad u, w \in \mathcal{V} \tag{3.6}
\end{equation*}
$$

which, in turn, induces the norm on $\mathcal{V}$ :

$$
\begin{equation*}
\|u\| \equiv \sqrt{(u, u)_{\Omega}} . \tag{3.7}
\end{equation*}
$$

The starting point for finite element methods is the so-called weak or variational formulation. A weak formulation of the linearized Euler equations on $\Omega$ is obtained by multiplying the LEE by a test function $w(\mathbf{x})$ and by integrating the result over the spatial domain $\Omega$ :

Weak formulation on $\Omega$ : For every $t$, find $\mathbf{u} \in \mathcal{U}^{5}$ such that:

$$
\begin{equation*}
(\mathbf{L}(\mathbf{u}), w)_{\Omega}=(\mathbf{s}, w)_{\Omega}, \quad \forall w \in \mathcal{V} \tag{3.8}
\end{equation*}
$$

where, for $\mathbf{v}(\mathbf{x})=\left(v_{1}(\mathbf{x}), \ldots, v_{5}(\mathbf{x})\right)^{T}$ :

$$
\begin{equation*}
(\mathbf{v}, w)_{\Omega}=\left(\left(v_{1}, w\right)_{\Omega}, \ldots,\left(v_{5}, w\right)_{\Omega}\right)^{T} \tag{3.9}
\end{equation*}
$$

The starting point for the DG method is to develop an appropriate weak formulation of the LEE on a partition of $\Omega$ into elements, which we denote by $T_{h}(\Omega)$. In the present study $T_{h}(\Omega)$ partitions $\Omega$ into non-overlapping tetrahedral elements $\Omega_{j}$ :

$$
\begin{equation*}
\bar{\Omega}=\bigcup_{j=1}^{N_{e}} \bar{\Omega}_{j} \tag{3.10}
\end{equation*}
$$

where $\bar{\Omega}=\Omega \cup \partial \Omega$ is the closure of $\Omega$ and where $\Omega_{j} \cap \Omega_{l}=\emptyset$ for $l \neq j$. A bounding surface of an element can only belong to at most two elements. When it belongs to one element the surface is part of the domain boundary $\partial \Omega$.

In the semi-discrete formulation we obtain an approximate solution by defining the approximation space $\mathcal{V}_{h} \subset \mathcal{V}$. In the DG method the space $\mathcal{V}_{h}$ is continuous within each element but may be discontinuous across element interfaces. The approximation space $\mathcal{V}_{h}$ is obtained by defining on each element $\Omega_{j} \in T_{h}$ the local space $\mathcal{P}_{p_{j}}\left(\Omega_{j}\right)$ of polynomials of degree $\leq p_{j}$.

The degree $p_{j}$ may vary from element to element, however, we use $p_{j}=p$ for all elements $j$. Let $\left\{b_{j 0}, b_{j 1}, \ldots, b_{j M}\right\}$ be a complete set which forms a local basis for the local space $\mathcal{P}_{p}\left(\Omega_{j}\right)$, so:

$$
\begin{equation*}
\mathcal{P}_{p}\left(\Omega_{j}\right)=\operatorname{span}\left\{b_{j 0}, b_{j 1}, \ldots, b_{j M}\right\} \tag{3.11}
\end{equation*}
$$

The basis functions are defined such that:

$$
\begin{align*}
& \bar{b}_{j k}(\mathbf{x})=0, \quad \mathbf{x} \notin \bar{\Omega}_{j},  \tag{3.12}\\
& b_{j k} \equiv \begin{cases}\bar{b}_{j k}(\mathbf{x}), & \mathbf{x} \in \Omega_{j}, \\
0, & \mathbf{x} \notin \Omega_{j}\end{cases} \tag{3.13}
\end{align*}
$$

The basis functions $\bar{b}_{j k}$ are continuous functions in $\bar{\Omega}_{j}$. The approximation space $\mathcal{V}_{h}$ is defined as:

$$
\begin{equation*}
\mathcal{V}_{h}(\Omega) \equiv \bigcup_{j=1}^{N_{e}} \mathcal{P}_{p}\left(\Omega_{j}\right) \tag{3.14}
\end{equation*}
$$

In the semi-discrete formulation we consider an approximation $\mathbf{u}_{h}(\mathrm{x}, t) \in \mathcal{V}_{h} \times \mathcal{T}^{5}$ of the solution $\mathbf{u}(\mathbf{x}, t)$ which is given by the linear representation:

$$
\begin{equation*}
\mathbf{u}_{h}(\mathbf{x}, t)=\sum_{j=1}^{N_{e}} \mathbf{v}_{j k}(t) b_{j k}(\mathbf{x}), \quad \mathbf{v}_{j k}(t) \in \mathcal{T}^{5} \tag{3.15}
\end{equation*}
$$

(note the summation over index $k$, see also section 3.1). So, within an element we have:

$$
\begin{equation*}
\left.\mathbf{u}_{h}(\mathbf{x}, t)\right|_{\Omega_{j}}=\mathbf{u}_{j}(\mathbf{x}, t) \equiv \mathbf{v}_{j k}(t) b_{j k}(\mathbf{x}), \quad \text { for } \mathbf{x} \in \Omega_{j}, \tag{3.16}
\end{equation*}
$$

where the coefficients $\mathbf{v}_{j k}(t)$ are the unknowns. From the definition of the basis functions it follows that $\mathbf{u}_{h}=0$ for $\mathbf{x} \in \partial \Omega_{j}$. The approximate solution $\mathbf{u}_{h}$, which is local in each element, is discontinuous across element boundaries. In the space-time DG approach the basis set contains both spatial and temporal functions, see for example van der Vegt \& van der Ven [82], [83].

We approximate the weak formulation of Eq.(3.8) by:
Discontinuous Galerkin approximation on $\Omega$ : For every $t$, find $\mathbf{u}_{h} \in \mathcal{V}_{h} \times \mathcal{T}^{5}$ such that:

$$
\begin{equation*}
\left(\mathbf{L}\left(\mathbf{u}_{h}\right), w_{h}\right)_{\Omega}=\left(\mathbf{s}, w_{h}\right)_{\Omega}, \quad \forall w_{h} \in \mathcal{V}_{h} \tag{3.17}
\end{equation*}
$$

Eq.(3.17) can be replaced by:

$$
\begin{equation*}
\left(\mathbf{L}\left(\mathbf{u}_{h}\right), b_{j m}\right)_{\Omega_{j}}=\left(\mathbf{s}, b_{j m}\right)_{\Omega_{j}}, \quad \forall m, j, \tag{3.18}
\end{equation*}
$$

which can be explained as follows; Eq.(3.17) holds for any function $w_{h} \in \mathcal{V}_{h}$, so since $b_{j m} \in \mathcal{V}_{h}$, it holds for $w_{h}=b_{j m}$. Subsequently, using the fact that $b_{j m}(\mathbf{x})=0$ for $\mathbf{x} \notin \Omega_{j}$ we arrive at Eq.(3.18). From Eq.(3.18) we retrieve Eq.(3.17) upon noting that any function $w_{h} \in \mathcal{V}_{h}$ can be written as a linear representation like Eq.(3.15). So, upon multiplying both sides of the Eq.(3.18) by the coefficient belonging to $b_{j k}$ and summing over both $j$ and $k$ and upon noting that the inner product is a linear operator, we arrive again at Eq.(3.17).

Using Eqs.(3.1) we evaluate Eq.(3.18) for every element $\Omega_{j} \in T_{h}$ :

$$
\begin{equation*}
\left(\frac{\partial \mathbf{u}_{h}}{\partial t}, b_{j m}\right)_{\Omega_{j}}+\left(\frac{\partial \mathbf{f}_{i}}{\partial x_{i}}, b_{j m}\right)_{\Omega_{j}}=\left(\mathbf{s}, b_{j m}\right)_{\Omega_{j}}, \quad \forall m, j . \tag{3.19}
\end{equation*}
$$

### 3.2.2 Numerical flux

Closer inspection of Eq.(3.19) reveals that the solution within an element only depends on information within that element, i.e. there is no "communication" between elements. Furthermore the global solution is, in general, discontinuous over an element interface.

Upon partial integration of the second term of Eq.(3.19) we obtain:

$$
\begin{equation*}
\left(\frac{\partial \mathbf{f}_{i}}{\partial x_{i}}, b_{j m}\right)_{\Omega_{j}}=-\left(\mathbf{f}_{i}, \frac{\partial b_{j m}}{\partial x_{i}}\right)_{\Omega_{j}}+\left(\frac{\partial \mathbf{f}_{i} b_{j m}}{\partial x_{i}}, 1\right)_{\Omega_{j}} \tag{3.20}
\end{equation*}
$$

Employing Gauss' theorem the last term can be written as:

$$
\begin{equation*}
\left(\frac{\partial \mathbf{f}_{i} b_{j m}}{\partial x_{i}}, 1\right)_{\Omega_{j}}=\left(\frac{\partial \mathbf{f}_{i} \bar{b}_{j m}}{\partial x_{i}}, 1\right)_{\Omega_{j}}=\int_{\partial \Omega_{j}} n_{i} \mathbf{f}_{i} \bar{b}_{j m} d \Gamma \tag{3.21}
\end{equation*}
$$

where $\mathbf{n}$ is the outward unit normal to the boundary.
In the interior of the mesh, each of the four bounding surfaces of each element connects two elements. So, by applying Eq.(3.21) we have two different values for the flux on each inter-element surface. On the common surface $\partial \Omega_{j l}=\partial \Omega_{j} \cap \partial \Omega_{l}$ between elements $\Omega_{j}$ and $\Omega_{l}$ we have:

$$
\begin{equation*}
\int_{\partial \Omega_{j l}} n_{i} \mathbf{f}_{i} \bar{b}_{l m} d \Gamma, \tag{3.22}
\end{equation*}
$$

where $I_{j}=\left\{k_{1}, k_{2}, k_{3}, k_{4}\right\}$ is the set of global indices, describing the direct neighborhood elements $\Omega_{k_{l}}$ of element $\Omega_{j}$ :

$$
\begin{align*}
& j \notin I_{j},  \tag{3.23}\\
& l \in I_{j} \quad \Longleftrightarrow \quad \partial \Omega_{j} \cap \partial \Omega_{l} \neq \emptyset, \quad j \neq l . \tag{3.24}
\end{align*}
$$

The common edges are denoted by $\partial \Omega_{j l}=\partial \Omega_{j} \cap \partial \Omega_{l}, l \in I_{j}$. The set $I_{j}$ has at most four elements. When one or more of the edges of the tetrahedron $\Omega_{j}$ belong to the domain boundary $\partial \Omega$, the set $I_{j}$ contains less than four elements, with a minimum of one.

The basis functions on the boundary induce the following linear representations for the approximate solution on the boundary:

$$
\begin{equation*}
\overline{\mathbf{u}}_{j}(\mathbf{x}, t) \equiv \mathbf{v}_{j k}(t) \bar{b}_{j k}(\mathbf{x}), \quad \mathbf{x} \in \partial \Omega_{j} \tag{3.25}
\end{equation*}
$$

To provide the crucial coupling and to handle the discontinuity at element interfaces, the boundary-normal flux, $\left(n_{j}\right)_{i} \mathbf{f}_{i}\left(\overline{\mathbf{u}}_{j}\right)$, is replaced by an approximate Riemann flux $\mathbf{h}\left(\overline{\mathbf{u}}_{j}, \overline{\mathbf{u}}_{l}, \mathbf{n}_{j}\right)$. In the linearized Euler equations also the coefficients of the matrices $A_{i}$, related to the mean or background flow, can vary with $\mathbf{x}$, i.e $A_{i}=A_{i}(\mathbf{x})$. In the current work, however, the mean flow properties have been approximated as piecewise constant per element:

$$
\begin{equation*}
A_{i}\left(\mathbf{x}_{j}\right)=A_{i}\left(\mathbf{x}_{j 0}\right)+\mathcal{O}(h), \quad A_{i}\left(\mathbf{x}_{j 0}\right) \equiv A_{i 0} \tag{3.26}
\end{equation*}
$$

where $\mathbf{x}_{j 0}$ denotes the location of the centroid of element $\Omega_{j}$ and where $h$ is a characteristic length scale of the element. On the interface between two elements we use the average of the two states:

$$
\begin{equation*}
\bar{A}^{n}=\frac{1}{2}\left(\left(A_{0}^{n}\right)^{L}+\left(A_{0}^{n}\right)^{R}\right), \quad A_{0}^{n}=n_{i} A_{i 0} \tag{3.27}
\end{equation*}
$$

These assumptions greatly simplify the method, i.e. the matrices $A_{i 0}$ and $\bar{A}^{n}$ can now be taken out of the integrations, but it does effect the global accuracy of the method.

The approximate Riemann flux must be both consistent

$$
\begin{equation*}
\mathbf{h}(\mathbf{u}, \mathbf{u}, \mathbf{n})=\overline{\mathbf{f}}^{n}(\mathbf{u}), \quad \overline{\mathbf{f}}^{n}(\mathbf{u}) \equiv \bar{A}^{n} \mathbf{u} \tag{3.28}
\end{equation*}
$$

and conservative:

$$
\begin{equation*}
\mathbf{h}\left(\mathbf{u}^{L}, \mathbf{u}^{R}, \mathbf{n}\right)=\mathbf{h}\left(\mathbf{u}^{R}, \mathbf{u}^{L},-\mathbf{n}\right) \tag{3.29}
\end{equation*}
$$

In the current work the local Lax-Friedrichs flux is used as approximate Riemann flux:

$$
\begin{equation*}
\mathbf{h}\left(\mathbf{u}^{L}, \mathbf{u}^{R}, \mathbf{n}\right)=\frac{1}{2}\left(\overline{\mathbf{f}}^{n}\left(\mathbf{u}^{L}\right)+\overline{\mathbf{f}}^{n}\left(\mathbf{u}^{R}\right)\right)-\frac{1}{2} \alpha\left(\mathbf{u}^{R}-\mathbf{u}^{L}\right), \tag{3.30}
\end{equation*}
$$

where $\alpha$ is larger in magnitude than the largest eigenvalue of the ( $5 \times 5$ )-matrix $\bar{A}^{n}$.
In Eq.(3.21) we now replace the normal flux by:

$$
\begin{equation*}
\left.\left(n_{j}\right)_{i} \mathbf{f}_{i}\left(\mathbf{u}_{j}\right)\right|_{\mathbf{x} \in \partial \Omega_{j l}}=\left.\mathbf{h}\left(\overline{\mathbf{u}}_{j}, \overline{\mathbf{u}}_{l}, \mathbf{n}_{j}\right)\right|_{\mathbf{x} \in \partial \Omega_{j l}}+\mathcal{O}\left(\overline{\mathbf{u}}_{l}-\overline{\mathbf{u}}_{j}\right) \tag{3.31}
\end{equation*}
$$

Together with the Lax-Friedrichs flux, the resulting set of equations can be written, in integral notation, as:

$$
\begin{align*}
\int_{\Omega_{j}} \frac{\partial \mathbf{u}_{h}}{\partial t} b_{j m} d \Omega-\int_{\Omega_{j}} \mathbf{f}_{i}\left(\mathbf{u}_{h}\right) \frac{\partial b_{j m}}{\partial x_{i}} d \Omega & +\sum_{l \in I_{j}} \int_{\partial \Omega_{j l}} \mathbf{h}\left(\overline{\mathbf{u}}_{j}, \overline{\mathbf{u}}_{l}, \mathbf{n}_{j}\right) \bar{b}_{j m} d \Gamma \\
& =\int_{\Omega_{j}} \mathbf{s} b_{j m} d \Omega, \quad \forall m, j . \tag{3.32}
\end{align*}
$$

Eq.(3.32) shows the extreme compactness of the method. Most higher-order methods need large stencils of elements in order to obtain higher order accuracy. In the DG method higherorder accuracy is obtained by representing the solution as a higher-order degree polynomial within each element. The fact that communication occurs through a surface integral means that the solution in any element $\Omega_{j}$ only depends on the solution within that element itself and the solutions of neighboring elements with which it has a surface in common. The extreme compactness of the method makes the method very well-suited for implementation on tetrahedral meshes, or for that matter on any unstructured mesh. Furthermore, the compact form of the DG method makes it well suited for implementation on parallel computer platforms.

### 3.2.3 Polynomial basis functions

We define the basis functions in the, so-called, master or reference element $\hat{\Omega}$. The local coordinates in the master element are given by $\boldsymbol{\xi}=(\xi, \eta, \zeta)^{T}$ and the coordinate system has its origin at the centroid of the tetrahedron. The physical coordinates $\mathbf{x}_{j}$ of element $\Omega_{j}$ are related to the computational coordinates of the master element by the invertible map:

$$
\begin{align*}
& \mathbf{x}^{j}: \quad \overline{\hat{\Omega}} \mapsto \bar{\Omega}_{j}, \\
& \mathbf{x}^{j}(\boldsymbol{\xi})=\mathbf{x}_{j 0}+J_{j} \boldsymbol{\xi}, \quad J_{j} \in \mathbb{R}^{3} \times \mathbb{R}^{3} . \tag{3.33}
\end{align*}
$$

Here $\mathbf{x}_{j 0}=\left(x_{0}, y_{0}, z_{0}\right)_{j}^{T}$ denotes the location of the centroid of element $\Omega_{j}$ (see also Fig.(3.1)), relative to the fixed coordinate system $\mathbf{x}=(x, y, z)^{T}$, defined for the whole domain $\Omega$ and $J_{j}$ is a non-singular ( $3 \times 3$ )-matrix with constant coefficients. The matrices $J_{j}$ are in general different for each elements. Figure 3.1 shows the mapping from the master element $\overline{\hat{\Omega}}$ onto an element $\bar{\Omega}_{j}$.


Figure 3.1: Mapping from master element $\bar{\Omega}$ onto $\bar{\Omega}_{j}$ and vise versa.

The map $\boldsymbol{\xi}^{j}$ is the inverse of $\mathbf{x}^{j}$ :

$$
\begin{align*}
& \boldsymbol{\xi}^{j}: \quad \bar{\Omega}_{j} \mapsto \overline{\hat{\Omega}} \\
& \boldsymbol{\xi}^{j}(\mathbf{x})=J_{j}^{-1}\left(\mathbf{x}-\mathbf{x}_{j 0}\right), \quad \Rightarrow \quad \boldsymbol{\xi}^{j}\left(\mathbf{x}^{j}(\boldsymbol{\xi})\right)=\boldsymbol{\xi} \tag{3.34}
\end{align*}
$$

Both the maps $\mathbf{x}^{j}$ and $\boldsymbol{\xi}^{j}$ are linear, but for a constant.
On $\bar{\Omega}$ we define a set of linearly independent polynomials $\left\{b_{k}(\xi, \eta, \zeta)\right\}$ of degree $\leq p$ :

$$
\begin{equation*}
\left\{b_{k}\right\}=\left\{\xi^{k_{1}} \eta^{k_{2}} \zeta^{k_{3}} \text { for } 0 \leq k_{1}+k_{2}+k_{3} \leq p\right\} \tag{3.35}
\end{equation*}
$$

The set $\left\{b_{k}\right\}$ is complete in the sense that it spans $\hat{\mathcal{P}}_{p}(\overline{\hat{\Omega}})$, the space of all polynomials on $\overline{\hat{\Omega}}$ with real coefficients and with a degree $\leq p$ :

$$
\begin{equation*}
\hat{\mathcal{P}}_{p}(\hat{\Omega})=\operatorname{span}\left\{b_{0}, b_{1}, \ldots, b_{M}\right\} \tag{3.36}
\end{equation*}
$$

The set $\left\{b_{k}(\xi, \eta, \zeta)\right\}$, which we will call the set of basis functions, induces a set of functions on $\bar{\Omega}_{j}$ by:

$$
\begin{equation*}
\bar{b}_{j k} \equiv b_{k}\left(\boldsymbol{\xi}^{j}(\mathbf{x})\right), \quad \mathbf{x} \in \bar{\Omega}_{j} \tag{3.37}
\end{equation*}
$$

The total number of basis functions, $M+1$, present in the sets $\left\{b_{k}\right\}$ and $\left\{b_{j k}\right\}$, depends on the degree $p$ of the polynomials and is given by the relation:

$$
\begin{equation*}
M(p, d)+1=\frac{1}{d!} \prod_{k=0}^{d}(p+k) \tag{3.38}
\end{equation*}
$$

where $d$ is the number of spatial dimensions, which in our case is equal to 3 . For example, for $p=1$ we have $M(1,3)=3$ and we have in total 4 basis functions: $1, \xi, \eta$ and $\zeta$.

Johnson and Pitkärata [45] prove that, when the basis functions are polynomials of degree less or equal to $p$, the rate of convergence of the DG method for general triangulations is at least $h^{p+\frac{1}{2}}$, where $h$ is a characteristic length scale of the elements. Atkins and Shu [6] mention that in most practical cases they observed a rate of convergence of $h^{p+1}$. Additionally, Lesaint and Raviart [53] and Richter [68] proved that the rate of convergence is of order $h^{p+1}$ on a Cartesian grid and on a semi-uniform triangular grid, respectively. Based on these observations we will call the method a $(p+1)^{t h}$-order method.

### 3.2.4 Quadrature-free evaluation of integrals

## Quadrature-free approach

In Eq.(3.32) volume and surface integrals have to be evaluated. Upon substitution of the linear representation of Eq.(3.16) into Eq.(3.32), it follows that the integration of the first term can easily be performed, since it only involves integration of products of the basis functions. The other terms are often evaluated employing numerical quadrature formulas of the required order (Cockburn et al. [19]). About numerical quadrature Atkins \& Shu [6] mention the following:
'The application of numerical quadrature is simple and straightforward, however, the associated computational costs are high. This because numerical quadrature is most efficient when the unknowns are stored at the quadrature points. In the DG method the unknowns are stored as expansion coefficients in the centroids of the elements and an order $M+1$ operations are needed for each quadrature point to obtain the required data to evaluate the numerical quadrature (Cockburn et al. [19]). Furthermore, optimal quadrature formulas of arbitrary order are not generally available for arbitrary element shapes. Tensor products of one-dimensional quadrature formulas can be used to integrate quadrilateral and hexahedral elements to any required degree. For general elements the only available quadratures are computed numerically for a limited range of order $p$. This has restricted most implementations of the DG method to quadrilateral, hexahedral or relatively low-order triangular elements."

In the quadrature-free approach, developed by Atkins \& Shu ([5]), quadrature rules do not have to be applied. Instead the fluxes and sources are, like the approximate solution, written as linear representations in terms of the basis functions:

$$
\begin{align*}
& \mathbf{f}_{i}(\mathbf{x}, t) \approx\left(\mathbf{f}_{i}\right)_{h}=\sum_{j=1}^{N_{e}}\left(\mathbf{f}_{i_{j k}} b_{j k}, \quad \forall k \in[0, N],\right.  \tag{3.39}\\
& \mathbf{s}(\mathbf{x}, t) \approx \mathbf{s}_{h}=\sum_{j=1}^{N_{e}} \mathbf{s}_{j k} b_{j k}, \quad \forall k \in[0, N], \tag{3.40}
\end{align*}
$$

where $N \geq M(p, d), M$ is given by Eq.(3.38). The quadrature-free approach is ideally suited for linear problems. When $\mathbf{f}_{i}$ (and or s) are linear functions of $\mathbf{u}$, the flux $\mathbf{f}_{i}\left(\mathbf{u}_{h}\right)$ is expanded in a natural way and $N=M$. When the flux is non-linear or linear but with non-constant coefficients, however, then the degree of the flux expansion has to be at least $p+1$ and hence $N>M$ (Atkins \& Shu [5], [6]). Atkins \& Lockard [4] have applied the quadrature-free approach to both linear and non-linear problems and performed time-accurate
simulations of acoustic propagation and scatter about full scale aircraft configurations using general unstructured grids.

As mentioned by Cockburn et al.[19] the problem of the quadrature-free approach is to obtain the number $N$ at which the expansion can be truncated such to ensure both accuracy and stability of the resulting DG method. Recently van der Ven \& van der Vegt [79] introduced the so-called Taylor quadrature rule, which is related to the quadrature-free approach of Atkins \& Shu, which has been described above. As in the quadrature-free approach, the flux is expanded in the basis functions, however, the coefficients are obtained from a direct Taylor expansion of the flux in the face-center. For linear fluxes this flux expansion is equal to the flux expansion in the quadrature-free approach (van der Ven \& van der Vegt [79]), however, for the non-linear Euler flux the expansion is different, i.e. only as many terms as the number of basis functions in the DG method are used. It is noted here that in [79], van der Ven \& van der Vegt consider the space-time DG discretization for the Euler equations with linear basis functions (basis functions of degree $\leq 1$ ). In van der Ven \& van der Vegt ([79]), the DG method with linear basis functions in combination with the Taylor quadrature rule resulted in a second-order local truncation error.

In the present work the flux is linear in $\mathbf{u}$. In most cases considered the mean flow is assumed uniform, from which it follows that the coefficients of the matrices $A_{i}$ are independent of $\mathbf{x}$ and the coefficient matrices can be taken out of the integrals. In the cases in which the coefficients of $A_{i}$, presented in Eq.(3.3), are functions of $\mathbf{x}$, we approximate them as piecewise constant, i.e. constant per element (see also section 3.2.2). In future work this approximation may be removed and the flux, with non-constant coefficient matrices $A_{i}$, may be expanded in the basis functions with $N>M$.

Next we will evaluate the integrals present in Eq.(3.32), employing the quadrature-free approach, and obtain the results in terms of a matrix vector product, where the matrices can be pre-computed.

## Evaluation of integrals

Employing the linear representations given by Eq.(3.15) and Eq.(3.40), the individual terms in Eq.(3.32) can be evaluated.

## - First integral:

The first integral of Eq.(3.32) is given by:

$$
\begin{equation*}
\int_{\Omega_{j}} \frac{\partial \mathbf{u}_{h}}{\partial t} b_{j m} d \Omega=\frac{d \mathbf{v}_{j k}}{d t} \int_{\Omega_{j}} b_{j k} b_{j m} d \Omega . \tag{3.41}
\end{equation*}
$$

Employing the inverse of the linear map $\mathbf{x}^{j}$, denoted by $\boldsymbol{\xi}^{j}$ and defined by Eq.(3.34), and the basis functions $b_{j k}(\mathbf{x})$, we evaluate the remaining integral as:

$$
\begin{equation*}
\int_{\Omega_{j}} b_{j k}(\mathbf{x}) b_{j m}(\mathbf{x}) d \Omega=\left|J_{j}\right| \int_{\hat{\Omega}} b_{k}(\boldsymbol{\xi}) b_{m}(\boldsymbol{\xi}) d \hat{\Omega} \tag{3.42}
\end{equation*}
$$

Introducing the notation:

$$
\begin{equation*}
M_{k m} \equiv \int_{\hat{\Omega}} b_{k} b_{m} d \hat{\Omega}, \tag{3.43}
\end{equation*}
$$

we obtain:

$$
\begin{equation*}
\int_{\Omega_{j}} \frac{\partial \mathbf{u}_{h}}{\partial t} b_{j m} d \Omega=\left|J_{j}\right| M_{k m} \frac{d \mathbf{v}_{j k}}{d t} . \tag{3.44}
\end{equation*}
$$

The matrix $M$ is commonly named the mass matrix. The name has been adopted from structural mechanics.

## - Second integral:

The second integral of Eq.(3.32) is given by:

$$
\begin{equation*}
\int_{\Omega_{j}} \mathbf{f}_{i}\left(\mathbf{u}_{h}\right) \frac{\partial b_{j m}}{\partial x_{i}} d \Omega=\int_{\Omega_{j}} A_{i} \mathbf{v}_{j k} b_{j k} \frac{\partial b_{j m}}{\partial x_{i}} d \Omega . \tag{3.45}
\end{equation*}
$$

Following the discussion above, the mean flow is approximated as piecewise constant per element $A_{i} \approx A_{i 0}$, see also Eq.(3.26). The matrix vector products $A_{i 0} \mathbf{v}_{j k}$ can now be taken out of the integral:

$$
\begin{equation*}
\int_{\Omega_{j}} A_{i 0} \mathbf{v}_{j k} b_{j k} \frac{\partial b_{m}}{\partial x_{i}} d \Omega=A_{i 0} \mathbf{v}_{j k} \int_{\Omega_{j}} b_{j k} \frac{\partial b_{j m}}{\partial x_{i}} d \Omega \tag{3.46}
\end{equation*}
$$

Employing the relation $\frac{\partial}{\partial x_{i}}=\frac{\partial \xi_{l}}{\partial x_{i}} \frac{\partial}{\partial \xi_{l}}$ and $\left(J_{j}\right)_{i l}=\frac{\partial x_{i}}{\partial \xi_{l}}$, integration in the master element results in:

$$
\begin{equation*}
\int_{\Omega_{j}} b_{j k} \frac{\partial b_{j m}}{\partial x_{i}} d \Omega=\left|J_{j}\right|\left(J_{j}^{-T}\right)_{i l} \int_{\hat{\Omega}} b_{k} \frac{\partial b_{m}}{\partial \xi_{l}} d \hat{\Omega} . \tag{3.47}
\end{equation*}
$$

Hence, we have:

$$
\begin{equation*}
\int_{\Omega_{j}} \mathbf{f}_{i}\left(\mathbf{u}_{h}\right) \frac{\partial b_{j m}}{\partial x_{i}} d \Omega=\left|J_{j}\right|\left\{A_{i 0}\left(J_{j}^{-T}\right)_{i l} \int_{\hat{\Omega}} b_{k} \frac{\partial b_{m}}{\partial \xi_{l}} d \hat{\Omega}\right\} \mathbf{v}_{j k} \tag{3.48}
\end{equation*}
$$

which, upon introdicing the notation:

$$
\begin{equation*}
F_{j k m} \equiv A_{i 0}\left(J_{j}^{-T}\right)_{i l} \int_{\hat{\Omega}} b_{k} \frac{\partial b_{m}}{\partial \xi_{l}} d \hat{\Omega} \tag{3.49}
\end{equation*}
$$

can be written as:

$$
\begin{equation*}
\int_{\Omega_{j}} \mathbf{f}_{i}\left(\mathbf{u}_{h}\right) \frac{\partial b_{j m}}{\partial x_{i}} d \Omega=\left|J_{j}\right| F_{j k m} \mathbf{v}_{j k} \tag{3.50}
\end{equation*}
$$

## - Third integral:

The third integral in Eq.(3.32) is (much) more complicated to evaluate. The integral is given by:

$$
\begin{align*}
\int_{\partial \Omega_{j l}} \mathbf{h}\left(\overline{\mathbf{u}}_{j}, \overline{\mathbf{u}}_{l}, \mathbf{n}_{j}\right) \bar{b}_{j m} d \Gamma=\int_{\partial \Omega_{j l}}\{ & \frac{1}{2}\left(\overline{\mathbf{f}}^{n}\left(\overline{\mathbf{u}}_{j}\right)+\overline{\mathbf{f}}^{n}\left(\overline{\mathbf{u}}_{l}\right)\right)- \\
& \left.\frac{1}{2} \alpha\left(\overline{\mathbf{u}}_{l}-\overline{\mathbf{u}}_{j}\right)\right\} \bar{b}_{j m} d \Gamma . \tag{3.51}
\end{align*}
$$

Employing Eq.(3.25) , the above equation can be written as:

$$
\begin{align*}
\int_{\partial \Omega_{j l}} \mathbf{h}\left(\overline{\mathbf{u}}_{j}, \overline{\mathbf{u}}_{l}, \mathbf{n}_{j}\right) \bar{b}_{j m} d \Gamma= & \frac{1}{2}\left[\bar{A}^{n}+\alpha I\right] \mathbf{v}_{j k} \int_{\partial \Omega_{j l}} \bar{b}_{j k} \bar{b}_{j m} d \Gamma+ \\
& \frac{1}{2}\left[\bar{A}^{n}-\alpha I\right] \mathbf{v}_{l k} \int_{\partial \Omega_{j l}} \bar{b}_{l k} \bar{b}_{j m} d \Gamma \tag{3.52}
\end{align*}
$$

where $I$ is the $(5 \times 5)$-identity matrix.
For convenience, the surface integrals of the right-hand-side of Eq.(3.52) will be evaluated on the equilateral reference triangle $\hat{\Gamma}$, which is shown in Fig.(3.2). The local coordinates of the equilateral reference triangle are given by $\hat{\boldsymbol{\xi}}=(\hat{\xi}, \hat{\eta})^{T}$ and the local coordinate system has its origin at the centroid of the triangle. The four surfaces of the reference element are also all equilateral triangles. The surface area of $\hat{\Gamma}$ is equal to the surface area of $\partial \hat{\Omega}_{s}$. The surface coordinates of $\partial \hat{\Omega}_{s}$, expressed in terms of the local element coordinates $(\xi, \eta, \zeta)$ of the reference element $\hat{\Omega}$, are related to the local surface coordinates of the equilateral reference triangle by the map:

$$
\begin{align*}
& \phi^{s}(\hat{\boldsymbol{\xi}}): \quad \hat{\Gamma} \mapsto \partial \hat{\Omega}_{s}, \\
& \phi^{s}(\hat{\boldsymbol{\xi}}) \equiv \boldsymbol{\xi}_{s 0}+\Phi_{s} \hat{\boldsymbol{\xi}}, \quad \Phi_{s} \in \mathbb{R}^{2} \times \mathbb{R}^{3}, \tag{3.53}
\end{align*}
$$

where $\boldsymbol{\xi}_{s 0}=\left(\xi_{0}, \eta_{0}, \zeta_{0}\right)_{s}^{T}$ represents the location of the centroid of the surface $\partial \hat{\Omega}_{s}$, expressed in terms of the local element coordinates of the reference element $\hat{\Omega}$ and where the index $s$ depends on the indices of elements $\Omega_{j}$ and $\Omega_{l}$ by:

$$
\begin{equation*}
s=\bar{s}(j, l) \tag{3.54}
\end{equation*}
$$

and where $\bar{s}(j, l)$ is such that

$$
\begin{equation*}
\left.\mathbf{x}^{j}\left(\phi^{s}(\hat{\Gamma})\right)\right)=\partial \Omega_{j l}, \quad s=\bar{s}(j, l) \tag{3.55}
\end{equation*}
$$

In Eq.(3.53) $\Phi_{s}$ is a $(2 \times 3)$-matrix. In addition, we introduce $\boldsymbol{\psi}^{s}$, which is the inverse map of $\phi^{s}$ :

$$
\begin{align*}
& \boldsymbol{\psi}^{s}(\boldsymbol{\xi}): \quad \partial \hat{\Omega}_{s} \mapsto \hat{\Gamma}, \\
& \boldsymbol{\psi}^{s}(\boldsymbol{\xi}) \equiv\left(\Phi_{s}^{T} \Phi_{s}\right)^{-1} \Phi_{s}^{T}\left(\boldsymbol{\xi}-\boldsymbol{\xi}_{s 0}\right) \quad \Rightarrow \quad \boldsymbol{\psi}^{s}\left(\boldsymbol{\phi}^{s}(\hat{\boldsymbol{\xi}})\right)=\hat{\boldsymbol{\xi}} \tag{3.56}
\end{align*}
$$



FIGURE 3.2: Mapping from equilateral reference triangle $\hat{\Omega}$ onto $\partial \hat{\Omega}_{s}$ and vise versa.
where it is required that $\operatorname{det}\left(\Phi_{s}^{T} \Phi_{s}\right) \neq 0$. It is noted that since the matrix $\Phi_{s}$ is not a square matrix it cannot be inverted. It can be shown that for all four maps from the faces $\partial \hat{\Omega}_{s}$ to the equilateral reference triangle $\operatorname{det}\left(\Phi_{s}^{T} \Phi_{s}\right) \neq 0$.

Employing the map $\phi^{s}(\hat{\boldsymbol{\xi}})$ we can write for the basis functions defined in the reference element $\hat{\Omega}$, Eq.(3.35):

$$
\begin{equation*}
b_{k}\left(\boldsymbol{\phi}^{s}(\hat{\boldsymbol{\xi}})\right)=\sum_{m=0}^{M(p, d-1)} T_{k m}^{s} \hat{b}_{m}(\hat{\boldsymbol{\xi}}) \tag{3.57}
\end{equation*}
$$

where the set $\left\{\hat{b}_{m}(\hat{\boldsymbol{\xi}})\right\}=\left\{1, \hat{\xi}, \hat{\eta}, \ldots, \hat{\eta}^{p}\right\}$ forms a basis for polynomials with degree less or equal to $p$ on $\hat{\Gamma} \subset \mathbb{R}^{2}$. The matrices $T^{s}$ are $(\hat{M} \times M)$-matrices, where

$$
\begin{equation*}
\hat{M}(p, d) \equiv M(p, d-1) \tag{3.58}
\end{equation*}
$$

and where $M(p, d)$ has been defined in Eq.(3.38). With this relation, Eq.(3.57), and the earlier obtained relation between the basis functions $\bar{b}_{j k}$ and $b_{k}$, Eq.(3.37), the first integral on the right-hand-side of Eq.(3.52) can be rewritten into an integral over basis functions $\hat{b}_{k}$ on $\hat{\Gamma}$ (with Einstein summation convention, except for index $j$ ):

$$
\begin{equation*}
\int_{\partial \Omega_{j l}} \bar{b}_{j k} \bar{b}_{j m} d \Gamma=\int_{\partial \hat{\Omega}_{l}} b_{k} b_{m}\left|J_{j}\right| d \boldsymbol{\xi}=\int_{\hat{\Gamma}} T_{k k^{\prime}}^{s} \hat{b}_{k^{\prime}} T_{m m^{\prime}}^{s} \hat{b}_{m^{\prime}}\left|J_{j}\right|\left|\hat{J}_{s}\right| d \hat{\Gamma}, \tag{3.59}
\end{equation*}
$$

where $\left|\hat{J}_{s}\right|=1$, because the surface area of the equilateral surfaces $\hat{\Gamma}$ and $\partial \hat{\Omega}_{s}$ are equal. So, we have:

$$
\begin{equation*}
\int_{\partial \Omega_{j l}} \bar{b}_{j k} \bar{b}_{j m} d \Gamma=\left|J_{j}\right| T_{k k^{\prime}}^{s} T_{m m^{\prime}}^{s} \int_{\hat{\Gamma}} \hat{b}_{k^{\prime}} \hat{b}_{m^{\prime}} d \hat{\Gamma}, \tag{3.60}
\end{equation*}
$$

The integral in Eq.(3.60) has to be evaluated only once, as a pre-processing step a numerical simulation. For the firts term on the right-hand-side of Eq.(3.52) we can now write:

$$
\begin{equation*}
\frac{1}{2}\left[\bar{A}^{n}+\alpha I\right] \mathbf{v}_{j k} \int_{\partial \Omega_{j l}} \bar{b}_{j k} \bar{b}_{j m} d \Gamma=\left|J_{j}\right|\left\{\frac{1}{2}\left[\bar{A}^{n}+\alpha I\right] T_{k k^{\prime}}^{s} T_{m m^{\prime}}^{s} \int_{\hat{\Gamma}} \hat{b}_{k^{\prime}} \hat{b}_{m^{\prime}} d \hat{\Gamma}\right\} \mathbf{v}_{j k} \tag{3.61}
\end{equation*}
$$

which, upon introducing the notation:

$$
\begin{equation*}
G_{j k m} \equiv \frac{1}{2}\left[\bar{A}^{n}+\alpha I\right] T_{k k^{\prime}}^{s} T_{m m^{\prime}}^{s} \int_{\hat{\Gamma}} \hat{b}_{k^{\prime}} \hat{b}_{m^{\prime}} d \hat{\Gamma} \tag{3.62}
\end{equation*}
$$

can be written as:

$$
\begin{equation*}
\frac{1}{2}\left[\bar{A}^{n}+\alpha I\right] \mathbf{v}_{j k} \int_{\partial \Omega_{j l}} \bar{b}_{j k} \bar{b}_{j m} d \Gamma=\left|J_{j}\right| G_{j k m} \mathbf{v}_{j k} \tag{3.63}
\end{equation*}
$$

The evaluation of the second integral on the right-hand-side of Eq.(3.52) involves integrating basis functions from both element $\Omega_{j}$ and $\Omega_{l}$. Although both can be mapped onto $\hat{\Gamma}$, the integration can not readily be performed because a common point on $\partial \Omega_{j l}$ is mapped onto different points in $\hat{\Gamma}$ when mapping this point from, respectively, element $\Omega_{j}$ and $\Omega_{l}$ onto $\hat{\Gamma}$. Assuming that a point on the surface $\mathbf{x} \in \partial \Omega_{j l}$ is part of element $\Omega_{j}$, it is mapped onto the point $\hat{\boldsymbol{\xi}}_{1}$ on the reference triangle $\hat{\Gamma}$. However, assuming that $\mathbf{x} \in \partial \Omega_{j l}$ is part of element $\Omega_{l}$ it is mapped onto $\hat{\boldsymbol{\xi}}_{2}$ on $\hat{\Gamma}$ :

$$
\begin{align*}
& \hat{\boldsymbol{\xi}}_{1}=\boldsymbol{\psi}^{s} \circ \boldsymbol{\xi}^{j}(\mathbf{x})=\boldsymbol{\psi}^{s}\left(\boldsymbol{\xi}^{j}(\mathbf{x})\right)  \tag{3.64}\\
& \hat{\boldsymbol{\xi}}_{2}=\boldsymbol{\psi}^{t} \circ \boldsymbol{\xi}^{l}(\mathbf{x})=\boldsymbol{\psi}^{t}\left(\boldsymbol{\xi}^{l}(\mathbf{x})\right) \tag{3.65}
\end{align*}
$$

where

$$
\begin{equation*}
s=\bar{s}(j, l) \quad \text { and } \quad t=\bar{s}(l, j) \tag{3.66}
\end{equation*}
$$

and where $\bar{s}(j, l)$ is as in Eq.(3.55). In general $\hat{\boldsymbol{\xi}}_{2} \neq \hat{\boldsymbol{\xi}}_{1}$. The above situation has been depicted in Fig.(3.3). From Eqs.(3.64) and (3.65) and the maps Eq.(3.53) and (3.56) $\hat{\boldsymbol{\xi}}_{2}$ can


Figure 3.3: Compound maps $\boldsymbol{\psi}^{s} \circ \boldsymbol{\xi}^{j}$ and $\boldsymbol{\psi}^{t} \circ \boldsymbol{\xi}^{l}$, mapping $\mathbf{x}$ onto $\hat{\boldsymbol{\xi}}_{1}$ and $\hat{\boldsymbol{\xi}}_{2}$, respectively.
be expressed in terms of $\hat{\boldsymbol{\xi}}_{1}$.

$$
\begin{align*}
\hat{\boldsymbol{\xi}}_{2} & =\boldsymbol{\psi}^{t} \circ \boldsymbol{\xi}^{l} \circ \mathbf{x}^{j} \circ \boldsymbol{\phi}^{s}\left(\hat{\boldsymbol{\xi}}_{1}\right)=\boldsymbol{\psi}^{t}\left(\boldsymbol{\xi}^{l}\left(\mathbf{x}^{j}\left(\boldsymbol{\phi}^{s}\left(\hat{\boldsymbol{\xi}}_{1}\right)\right)\right)\right),  \tag{3.67}\\
& =\hat{N}(\theta, \mu) \hat{\boldsymbol{\xi}}_{1} \tag{3.68}
\end{align*}
$$

where the matrix $\hat{N}$ is given by:

$$
\hat{N}(\theta, \mu)=\left[\begin{array}{cc}
\mu \cos (\theta) & -\sin (\theta)  \tag{3.69}\\
\mu \sin (\theta) & \cos (\theta)
\end{array}\right]
$$

and where $\theta=0$ or $\theta= \pm \frac{2}{3} \pi$ and $\mu= \pm 1$. The relation presented in Eq.(3.67) describes the path, which can followed step by step in Fig.(3.3); First $\hat{\boldsymbol{\xi}}_{1}$ is mapped onto a point on $\partial \hat{\Omega}_{s}$ in the reference element $\overline{\hat{\Omega}}$ via the map $\phi^{s}$, next this point on $\partial \hat{\Omega}_{s}$ is mapped onto $\mathbf{x}$ in $\partial \Omega_{j l}$ via the map $\mathbf{x}^{j}$. Now we have expressed $\mathbf{x}$ in terms of $\hat{\boldsymbol{\xi}}_{1}$. Subsequently $\mathbf{x}\left(\hat{\boldsymbol{\xi}}_{1}\right)$ is mapped via $\boldsymbol{\xi}^{l}$ and $\boldsymbol{\psi}^{t}$ to $\hat{\boldsymbol{\xi}}_{2}$ in $\hat{\Gamma}$.

The basis functions evaluated for $\mathbf{x}$ can now be expressed in terms of the basis functions $\hat{b}_{k}$ :

$$
\begin{align*}
\bar{b}_{l k}(\mathbf{x})=b_{k}\left(\boldsymbol{\xi}^{l}(\mathbf{x})\right) & =T_{k k^{\prime}}^{t} \hat{b}_{k^{\prime}}\left(\boldsymbol{\psi}^{t}\left(\boldsymbol{\xi}^{l}(\mathbf{x})\right)\right)= \\
& =T_{k k^{\prime}}^{t} \hat{b}_{k^{\prime}}\left(\boldsymbol{\psi}^{t}\left(\boldsymbol{\xi}^{l}\left(\mathbf{x}^{j}\left(\boldsymbol{\phi}^{s}(\hat{\boldsymbol{\xi}})\right)\right)\right)\right) \\
& =T_{k k^{\prime}}^{t} \hat{b}_{k^{\prime}}(\hat{N} \hat{\boldsymbol{\xi}})  \tag{3.70}\\
\bar{b}_{j m}(\mathbf{x})=b_{m}\left(\boldsymbol{\xi}^{j}(\mathbf{x})\right) & =T_{m m^{\prime}}^{s} \hat{b}_{m^{\prime}}\left(\boldsymbol{\psi}^{s}\left(\boldsymbol{\xi}^{j}(\mathbf{x})\right)\right) \\
& =T_{m m^{\prime}}^{s} \hat{b}_{m^{\prime}}(\hat{\boldsymbol{\xi}}) \tag{3.71}
\end{align*}
$$

For the second integral on the right-hand-side of Eq.(3.52) we now obtain:

$$
\begin{equation*}
\int_{\partial \Omega_{j l}} \bar{b}_{l k}(\mathbf{x}) \bar{b}_{j m}(\mathbf{x}) d \Gamma=\left|J_{j}\right| T_{k k^{\prime}}^{t} T_{m m^{\prime}}^{s} \int_{\hat{\Gamma}} \hat{b}_{k^{\prime}}(\hat{N} \hat{\boldsymbol{\xi}}) \hat{b}_{m^{\prime}}(\hat{\boldsymbol{\xi}}) d \hat{\Gamma} . \tag{3.72}
\end{equation*}
$$

Employing Eq.(3.69), the basis functions $\hat{b}_{k}(\hat{N} \hat{\boldsymbol{\xi}})$ can be written as:

$$
\begin{equation*}
\hat{b}_{k}(\hat{N} \hat{\boldsymbol{\xi}}) \equiv N_{k m} \hat{b}_{m}(\hat{\boldsymbol{\xi}}), \tag{3.73}
\end{equation*}
$$

and we can write for Eq.(3.72):

$$
\begin{equation*}
\int_{\partial \Omega_{j l}} \bar{b}_{l k}(\mathbf{x}) \bar{b}_{j m}(\mathbf{x}) d \Gamma=\left|J_{j}\right| T_{k k^{\prime}}^{t} T_{m m^{\prime}}^{s} N_{k^{\prime} n^{\prime}} \int_{\hat{\Gamma}} \hat{b}_{n^{\prime}} \hat{b}_{m^{\prime}} d \hat{\Gamma} \tag{3.74}
\end{equation*}
$$

Hence, we have obtained for the second part of the right-hand-side of equation Eq.(3.52):

$$
\begin{array}{r}
\frac{1}{2}\left[\bar{A}^{n}-\alpha I\right] \mathbf{v}_{l k} \int_{\partial \Omega_{j l}} \bar{b}_{l k} \bar{b}_{j m} d \Gamma= \\
\left|J_{j}\right|\left\{\frac{1}{2}\left[\bar{A}^{n}-\alpha I\right] T_{k k^{\prime}}^{t} T_{m m^{\prime}}^{s}\right.  \tag{3.75}\\
\\
\left.N_{k^{\prime} n^{\prime}} \int_{\hat{\Gamma}} \hat{b}_{n^{\prime}} \hat{b}_{m^{\prime}} d \hat{\Gamma}\right\} \mathbf{v}_{l k}
\end{array}
$$

which, upon introducing the notation:

$$
\begin{equation*}
H_{j k m}^{l} \equiv \frac{1}{2}\left[\bar{A}^{n}-\alpha I\right] T_{k k^{\prime}}^{t} T_{m m^{\prime}}^{s} N_{k^{\prime} n^{\prime}} \int_{\hat{\Gamma}} \hat{b}_{n^{\prime}} \hat{b}_{m^{\prime}} d \hat{\Gamma} \tag{3.76}
\end{equation*}
$$

can be written as:

$$
\begin{equation*}
\frac{1}{2}\left[\bar{A}^{n}-\alpha I\right] \mathbf{v}_{l k} \int_{\partial \Omega_{j l}} \bar{b}_{l k} \bar{b}_{j m} d \Gamma=\left|J_{j}\right| H_{j k m}^{l} \mathbf{v}_{l k} \tag{3.77}
\end{equation*}
$$

where we have to sum over the index $l$.
In summary, we have obtain for the third term given by Eq.(3.51), employing Eq.(3.52) and the results of Eq.(3.63) and Eq.(3.77):

$$
\begin{equation*}
\int_{\partial \Omega_{j l}} \mathbf{h}\left(\overline{\mathbf{u}}_{j}, \overline{\mathbf{u}}_{l}, \mathbf{n}_{j}\right) \bar{b}_{j m} d \Gamma=\left|J_{j}\right| G_{j k m} \mathbf{v}_{j k}+\left|J_{j}\right| H_{j k m}^{l} \mathbf{v}_{l k}, \tag{3.78}
\end{equation*}
$$

where we have to sum over $l$.

## - Fourth term:

The fourth term of Eq.(3.32) is given by:

$$
\begin{equation*}
\left(\mathbf{s}_{h}, b_{j m}\right)_{\Omega_{j}}=\left(b_{j k}, b_{j m}\right)_{\Omega_{j}} \mathbf{s}_{j k} . \tag{3.79}
\end{equation*}
$$

Clearly the integral which has to be evaluated here is exactly the same as the one which had to be evaluated for the first term. Employing Eq.(3.43) we therefore simply obtain:

$$
\begin{equation*}
\left(\mathbf{s}_{h}, b_{j m}\right)_{\Omega_{j}}=\left|J_{j}\right| M_{k m} \mathbf{s}_{j k} . \tag{3.80}
\end{equation*}
$$

### 3.2.5 Resulting system of equations

In the preceding section, section 3.2.4 we performed all the required integrations of Eq.(3.32). Combining the results of Eqs.(3.44), (3.50), (3.78) and (3.80) we obtain for Eq.(3.32) $\forall \Omega_{j} \in$ $T_{h}, \forall m \in[0, M(p, d)]:$

$$
\begin{equation*}
\left|J_{j}\right| M_{k m} \frac{d \mathbf{v}_{j k}}{d t}+\left|J_{j}\right|\left\{-G_{j m k}+F_{j m k}\right\} \mathbf{v}_{j k}+\left|J_{j}\right| H_{j m k}^{l} \mathbf{v}_{l k}=\left|J_{j}\right| M_{k m} \mathbf{s}_{k} \tag{3.81}
\end{equation*}
$$

where we have to sum over $l$. Hence, the determinant $\left|J_{j}\right|$ can be divided out of the equation, to result in:

$$
\begin{equation*}
M_{k m} \frac{d \mathbf{v}_{j k}}{d t}+\left\{-F_{j k m}+G_{j k m}\right\} \mathbf{v}_{j k}+H_{j m k}^{l} \mathbf{v}_{l k}=M_{k m} \mathbf{s}_{k} \tag{3.82}
\end{equation*}
$$

For the sake of completeness, the matrices $M_{k m}, F_{j k m}, G_{j k m}$ and $H_{j m k}^{l}$ are given by (Eqs.(3.43), (3.49), (3.62) and (3.76), respectively):

$$
\begin{aligned}
& M_{k m} \equiv \int_{\hat{\Omega}} b_{k} b_{m} d \hat{\Omega} \\
& F_{j k m} \equiv A_{i 0}\left(J_{j}^{-T}\right)_{i l} \int_{\hat{\Omega}} b_{k} \frac{\partial b_{m}}{\partial \xi_{l}} d \hat{\Omega} \\
& G_{j k m} \equiv T_{k k^{\prime}}^{s} T_{m m^{\prime}}^{s} \int_{\hat{\Gamma}} \hat{b}_{k^{\prime}} \hat{b}_{m^{\prime}} d \hat{\Gamma} \mathbf{v}_{j k} \\
& H_{j k m}^{l} \equiv \frac{1}{2}\left[\bar{A}^{n}-\alpha I\right] T_{k k^{\prime}}^{t} T_{m m^{\prime}}^{s} N_{k^{\prime} n^{\prime}} \int_{\hat{\Gamma}} \hat{b}_{n^{\prime}} \hat{b}_{m^{\prime}} d \hat{\Gamma}
\end{aligned}
$$

where (Eq.(3.55)):

$$
\left.s=\bar{s}(j, l), \quad \mathbf{x}^{j}\left(\phi^{s}(\hat{\Gamma})\right)\right)=\partial \Omega_{j l}
$$

### 3.3 Boundary conditions

An important feature of the DG method is that the approximate Riemann flux is the only mechanism through which an element communicates with its environment (direct neighbor or domain boundary), regardless of whether the bounding surface of the element is in the interior of the domain or coincides with the domain boundary ([7]). This makes that the system presented in Eq.(3.82) is also applicable in elements adjacent to the domain boundary. In contrast, most higher-order finite-difference and finite-volume methods, however, require a modification of the discretization stencils at points near the domain boundary. With increasing order, the modifications are needed for an increasing number of points, counted from the domain boundary inwards. The modification requirers special attention because it might adversely affect the accuracy and might even give rise to instabilities. Employing the DG method complications such as these are avoided.

The boundary conditions can be imposed by either providing the desired solution $\overline{\mathbf{u}}_{l}$ at the exterior side of the boundary, to be used in the approximate Riemann flux in Eq.(3.32):

$$
\begin{equation*}
\int_{\partial \Omega_{j l}} \mathbf{h}\left(\overline{\mathbf{u}}_{j}, \overline{\mathbf{u}}_{l}, \mathbf{n}_{j}\right) \bar{b}_{j m} d \Gamma \tag{3.83}
\end{equation*}
$$

or by reformulating the boundary normal flux, subjected to the specified boundary condition, such that only part of the interior data are needed ([7]). Since we usually do not have an exact solution at the exterior, we will specify the boundary condition by reformulating the boundary normal flux. In section 3.2.2 the integral involving the approximate Riemann flux, given by Eq.(3.83), replaced the surface integral of Eq.(3.22) and was introduced to provide coupling. In this section we will apply the different boundary conditions to boundary $\partial \Omega_{j b} \cap \partial \Omega$, where $\partial \Omega_{j b} \in \partial \Omega, \partial \Omega_{j b} \in \partial \Omega$, and reformulate Eq.(3.22):

$$
\begin{equation*}
\int_{\partial \Omega_{j b}} n_{i} \mathbf{f}_{i}\left(\overline{\mathbf{u}}_{j}\right) \bar{b}_{j m} d \Gamma \tag{3.84}
\end{equation*}
$$

where

$$
\begin{equation*}
n_{i} \mathbf{f}_{i}\left(\overline{\mathbf{u}}_{j}\right)=n_{i} A_{i 0} \overline{\mathbf{u}}_{j}=A_{0}^{n} \overline{\mathbf{u}}_{j}, \quad A_{0}^{n} \equiv n_{i} A_{i 0} \tag{3.85}
\end{equation*}
$$

and where (Eq.(3.25)):

$$
\overline{\mathbf{u}}_{j}(\mathbf{x}, t)=\mathbf{v}_{j k}(t) \bar{b}_{j k}(\mathbf{x}), \quad \mathbf{x} \in \partial \Omega_{j}
$$

Hence, we have:

$$
\begin{equation*}
\int_{\partial \Omega_{j b}} n_{i} \mathbf{f}_{i}\left(\overline{\mathbf{u}}_{j}\right) \bar{b}_{j m} d \Gamma=A_{0}^{n} \mathbf{v}_{j k} \int_{\partial \Omega_{j b}} \bar{b}_{j k} \bar{b}_{j m} d \Gamma \tag{3.86}
\end{equation*}
$$

In analogy with section 3.2.4, the integration will be performed employing the equilateral reference triangle $\hat{\Gamma}$. The notation which will be used here has been introduced in section
3.2.4. Employing the result of Eq.(3.60), we recast Eq.(3.86) into an integral over the equilateral reference triangle:

$$
\begin{equation*}
\int_{\partial \Omega_{j b}} n_{i} \mathbf{f}_{i}\left(\overline{\mathbf{u}}_{j}\right) \bar{b}_{j m} d \Gamma=A_{0}^{n} \mathbf{v}_{j k}\left|J_{j}\right| T_{k k^{\prime}}^{s} T_{m m^{\prime}}^{s} \int_{\hat{\Gamma}} \hat{b}_{k^{\prime}} \hat{b}_{m^{\prime}} d \hat{\Gamma} \tag{3.87}
\end{equation*}
$$

where $s$ is such that $\left.\mathbf{x}^{j}\left(\phi^{s}(\hat{\Gamma})\right)\right)=\partial \Omega_{j b}$. Upon introducing the notation, which is closely related to the notation introduced in Eq.(3.62):

$$
\begin{equation*}
G_{j k m}^{b} \equiv A_{0}^{n} T_{k k^{\prime}}^{s} T_{m m^{\prime}}^{s} \int_{\hat{\Gamma}} \hat{b}_{k^{\prime}} \hat{b}_{m^{\prime}} d \hat{\Gamma} \tag{3.88}
\end{equation*}
$$

we obtain

$$
\begin{equation*}
\int_{\partial \Omega_{j b}} n_{i} \mathbf{f}_{i}\left(\overline{\mathbf{u}}_{j}\right) \bar{b}_{j m} d \Gamma=G_{j k m}^{b} \mathbf{v}_{j k} \tag{3.89}
\end{equation*}
$$

The unit normal vector $\mathbf{n}$ to the boundary $\partial \Omega_{j b}$, which we will use frequently in this section, is derived from the unit normal vector to the boundary $\partial \hat{\Omega}_{s}$ in the reference element $\tau_{s}$, employing the relation:

$$
\begin{equation*}
\mathbf{n}=\frac{J_{j}^{-T} \boldsymbol{\tau}_{s}}{\left\|J_{j}^{-T} \boldsymbol{\tau}_{s}\right\|} \tag{3.90}
\end{equation*}
$$

It is noted that $\mathbf{n}$ and $\tau_{s}$ are constants on $\partial \Omega_{j b}$ and $\partial \hat{\Omega}_{s}$, respectively.
In the subsequent (sub-)sections the characteristics-based non-reflecting boundary condition for $\partial \Omega_{j b} \in \partial \Omega^{N R}$, solid wall boundary condition for $\partial \Omega_{j b} \in \partial \Omega^{S W}$ and vibrating wall boundary condition for $\partial \Omega_{j b} \in \partial \Omega^{V W}$, will be presented.

### 3.3.1 Characteristics-based non-reflecting boundary condition

We apply non-reflecting boundary conditions when we want to ensure that waves that are leaving the domain can do so, as if the boundary does not exist. The simplest form of non-reflecting inflow and outflow boundary conditions is obtained by splitting the boundary normal flux into characteristic components according to whether their associated wave is entering or leaving the domain. The Euler and the linearized Euler equations are hyperbolic in nature and can therefore be transformed into a characteristic form which facilitates the boundary treatment.

The ( $5 \times 5$ )- matrix $A_{0}^{n}$, defined in Eq.(3.27), has 5 eigenvalues $\lambda_{1}, \ldots, \lambda_{5}$, the corresponding eigenvectors $\mathbf{r}_{1}, \ldots, \mathbf{r}_{5}$ can be shown to be linearly independent. Now we can write ([64]):

$$
\begin{equation*}
A_{0}^{n}=R \Lambda R^{-1}, \tag{3.91}
\end{equation*}
$$

where

$$
\begin{equation*}
\Lambda=\operatorname{diag}\left(\lambda_{1}, \ldots, \lambda_{5}\right), \quad R=\left[\mathbf{r}_{1}, \ldots, \mathbf{r}_{5}\right] . \tag{3.92}
\end{equation*}
$$

We define:

$$
\begin{align*}
& \Lambda^{+}=\operatorname{diag}\left(\max \left(0, \lambda_{1}\right), \ldots, \max \left(0, \lambda_{5}\right)\right),  \tag{3.93}\\
& \Lambda^{-}=\operatorname{diag}\left(\min \left(0, \lambda_{1}\right), \ldots, \min \left(0, \lambda_{5}\right)\right), \tag{3.94}
\end{align*}
$$

and note that:

$$
\begin{equation*}
\Lambda^{+}+\Lambda^{-}=\Lambda \tag{3.95}
\end{equation*}
$$

such that:

$$
\begin{equation*}
A_{0}^{n} \overline{\mathbf{u}}_{j}=\left(R \Lambda^{+} R^{-1}\right) \overline{\mathbf{u}}_{j}+\left(R \Lambda^{-} R^{-1}\right) \overline{\mathbf{w}} \tag{3.96}
\end{equation*}
$$

where $\overline{\mathbf{w}}$ can be used to describe an incoming wave. Usually we set $\overline{\mathbf{w}}=\mathbf{0}$.
Setting $\overline{\mathbf{w}}=\mathbf{0}$, we replace $A_{0}^{n}$ in Eq.(3.88) by $R \Lambda^{+} R^{-1}$. Upon introducing the notation:

$$
\begin{equation*}
G_{j k m}^{N R} \equiv\left(R \Lambda^{+} R^{-1}\right) T_{k k^{\prime}}^{s} T_{m m^{\prime}}^{s} \int_{\hat{\Gamma}} \hat{b}_{k^{\prime}} \hat{b}_{m^{\prime}} d \hat{\Gamma} \tag{3.97}
\end{equation*}
$$

we obtain for the characteristics-based non-reflecting boundary condition:

$$
\begin{equation*}
\int_{\partial \Omega_{j b}} n_{i} \mathbf{f}_{i}\left(\overline{\mathbf{u}}_{j}\right) \bar{b}_{j m} d \Gamma=G_{j k m}^{N R} \mathbf{v}_{j k}, \quad \partial \Omega_{j b} \in \partial \Omega^{N R} \subset \partial \Omega . \tag{3.98}
\end{equation*}
$$

### 3.3.2 Solid wall

The solid-wall condition states that no flow passes through the boundary. Assuming that the mean flow satisfies this condition, we have to set the normal velocity perturbation equal to zero in order to implement this condition .

In vector notation, a vector $\mathbf{V}$ can be modified to have zero normal component, relative to the normal $\mathbf{n}$, by replacing it by the vector $\mathbf{W}$ :

$$
\begin{equation*}
\mathbf{W}=\mathbf{V}-(\mathbf{V} \cdot \mathbf{n}) \mathbf{n}, \quad \mathbf{W} \cdot \mathbf{n}=0 \tag{3.99}
\end{equation*}
$$

The three components of the velocity perturbation are given by the second, third and fourth component of $\overline{\mathbf{u}}_{j}$. Hence, to implement the solid-wall boundary condition for the velocity perturbation, we have to impose:

$$
\begin{equation*}
\bar{u}_{j}^{m+1} n_{m}=0, \quad m=1,2,3, \tag{3.100}
\end{equation*}
$$

where $\bar{u}_{j}^{m}$ denotes the $m^{t h}$-component of the solution vector $\overline{\mathbf{u}}_{j}$. Because the boundary normal vector is constant over the surface $\partial \Omega_{j b}$, the solid-wall boundary condition can be applied to Eq.(3.89) by replacing $\mathbf{v}_{j}$ by $\tilde{\mathbf{v}}_{j}$, where the second, third and fourth component are given by (in analogy with Eq.(3.99)):

$$
\begin{equation*}
\tilde{v}_{j k}^{m+1}=v_{j k}^{m+1}-\left(v_{j k}^{l+1} n_{l}\right) n_{m}, \quad m, l=1,2,3, \quad \forall k \tag{3.101}
\end{equation*}
$$

The first and fifth component of $\mathbf{v}_{j}$ remain unchanged. For the solid-wall boundary condition we finally obtain:

$$
\begin{equation*}
\int_{\partial \Omega_{j b}} n_{i} \mathbf{f}_{i}\left(\overline{\mathbf{u}}_{j}\right) \bar{b}_{j m} d \Gamma=G_{j k m}^{b} \tilde{\mathbf{v}}_{j k}, \quad \partial \Omega_{j b} \in \partial \Omega^{S W} \subset \partial \Omega . \tag{3.102}
\end{equation*}
$$

### 3.3.3 Vibrating wall

When considering a vibrating wall it is assumed that the surface-vibration amplitude is small compared with a representative acoustic wavelength and representative dimensions describing the surface. Assuming that the vibrating wall condition, i.e. the wall displacement, is introduced to the linearized Euker equations by a normal velocity boundary condition $\mathbf{v}_{\text {wall }}$ (as we will do in chapter 6), the boundary condition can be imposed by simply modifying Eq.(3.101):

$$
\begin{equation*}
\tilde{v}_{j k}^{m+1}=v_{j k}^{m+1}-\left(v_{j k}^{l+1} n_{l}+\left(v_{w a l l}\right)_{j k}\right) n_{m}, \quad m, l=1,2,3, \quad \forall k, \tag{3.103}
\end{equation*}
$$

where $\left(v_{\text {wall }}\right)_{j k}$ can be obtained by projecting $\mathbf{v}_{\text {wall }}$ onto the basis set $\left\{\hat{b}_{k}\right\}$. The coefficients can than be obtained from solving:

$$
\begin{equation*}
\int_{\hat{\Gamma}} \mathbf{v}_{\text {wall }} \hat{b}_{m} d \hat{\Gamma}=\left(v_{\text {wall }}\right)_{j k} \int_{\hat{\Gamma}} \hat{b}_{k} \hat{b}_{m} d \hat{\Gamma} \tag{3.104}
\end{equation*}
$$

Although $\mathbf{v}_{\text {wall }}$ is a known function of $\hat{\xi}$ and $\hat{\eta}$ in the entire surface, evaluation of the integrals, presented at the left-hand-side of Eq.(3.104) might still be cumbersome. As an alternative, the integrals can be evaluated employing numerical quadrature or the projection coefficients can be obtained by expanding $\mathbf{v}_{\text {wall }}$ in terms of the local basis functions $\hat{b}_{k}$.

### 3.4 Initial conditions

Let us assume that a computation is initiated at $t=0$ by imposing the initial condition given by $\mathbf{u}(\mathbf{x}, 0)=\phi(\mathbf{x})$.

The initial condition can be implemented by projection of $\phi(\mathbf{x})$ onto the approximation space $\mathcal{V}_{h}$. Let us denote the projection of $\phi$ onto the approximation space $\mathcal{V}_{h}$ by $\phi_{h}$, then:

$$
\begin{equation*}
\phi_{h}-\phi \quad \in \quad \mathcal{V}_{h}^{\perp} \tag{3.105}
\end{equation*}
$$

where $\mathcal{V}_{h}^{\perp}$ is the orthogonal complement of $\mathcal{V}_{h}$ and:

$$
\begin{equation*}
\left(\phi_{h}-\phi, b_{j m}\right)_{\Omega_{j}}=0 \Rightarrow\left(\phi, b_{j m}\right)_{\Omega_{j}}=\left(\phi_{h}, b_{j m}\right)_{\Omega_{j}}, \quad \forall m, j . \tag{3.106}
\end{equation*}
$$

Because $\phi_{h} \in \mathcal{V}_{h}$, we can write $\phi_{h}=\phi_{j k} b_{j k}$ :

$$
\begin{equation*}
\left(\phi, b_{j m}\right)_{\Omega_{j}}=\phi_{j k}\left(b_{j k}, b_{j m}\right)_{\Omega_{j}}, \quad \forall m, j . \tag{3.107}
\end{equation*}
$$

The integrations can be performed in the reference element $\hat{\Omega}$, employing the map $\mathbf{x}^{j}$ (Eq.(3.33)):

$$
\begin{equation*}
\left(\phi\left(\mathbf{x}^{j}(\boldsymbol{\xi})\right), b_{m}\right)_{\hat{\Omega}}=\phi_{j k}\left(b_{k}, b_{m}\right)_{\hat{\Omega}}, \quad \forall m, j . \tag{3.108}
\end{equation*}
$$

Although $\phi$ is a known function of $\boldsymbol{\xi}$ in the entire master element, the integrals presented at the left-hand-side of Eq.(3.108) might be difficult to evaluate. As an alternative, the integrals can be evaluated employing numerical quadrature or the projection coefficients can be obtained by expanding $\phi$ in terms of the basis functions $b_{k}$.

### 3.5 Runge-Kutta time integration

The time integration is performed applying the low-storage Runga-Kutta algorithm. Upon writing the system, presented in Eq.(3.82), for each element $\Omega_{j} \in T_{h}$ as:

$$
\begin{equation*}
\frac{d \mathbf{V}_{j}}{d t}=K \mathbf{V}_{j} \tag{3.109}
\end{equation*}
$$

In chapter 4 it will be shown that the set of equations presented in Eq.(3.82) can be written in the above form. The matrix $K$ is used as black-box, describing the right-hand-side of Eq.(3.82). It is comprised of the contributions of element $\Omega_{j}$ and its (at most) four direct neighbours, $\Omega_{l}, l \in I_{j}$. The solution at time $t+\Delta t$ is obtained from the solution at time $t$, $\forall \Omega_{j} \in T_{h}$, employing the following algorithm for the $N$-stage Runga-Kutta time integration:

$$
\begin{align*}
\mathbf{V}_{j}^{n, 0} & =\mathbf{V}_{j}^{n} \\
\mathbf{V}_{j}^{n, k} & =\mathbf{V}_{j}^{n}+\gamma_{k} \Delta t K \mathbf{V}_{j}^{n, k-1}, \quad k=1, \ldots, N, \\
\mathbf{V}_{j}^{n+1} & =\mathbf{V}_{j}^{n, N} \tag{3.110}
\end{align*}
$$

The coefficients $\gamma_{k}$ can be chosen such to obtain the required accuracy in time, but see also chapter 4.

# WAVE-PROPAGATION AND STABILITY ANALYSIS 

### 4.1 Introduction

Algorithms applied for aeroacoustic purposes require assessment of their numerical dispersion and dissipation, since these must be sufficiently low to accurately simulate the propagation of aeroacoustic information. The goal of this chapter is to get insight into the wavepropagation properties of the Discontinuous Galerkin method, which has been presented in the preceding chapter, chapter 3. The analysis will be performed for the algorithm obtained from applying the Discontinuous Galerkin method to discretize one-dimensional model problem in space, resulting in the semi-discrete algorithm. For the time discretization different algorithms will be considered: Euler explicit, Euler implicit and the multi-stage Runge-Kutta time-integration algorithms. Additionally, the stability of the fully-discrete algorithm will be investigated. The analysis is conducted assuming that the basis functions, used in the Discontinuous Galerkin method, are of degree $\leq p$, where $p$ can be set to any desired positive integer.

In contrast to studies on convergence rates, fewer studies have been presented on the wave propagation properties of the Discontinuous Galerkin method. One of the main contributors in the field of wave propagation analysis is most likely Hu. In [42], Hu and Atkins present a short overview of the studies presented on the wave propagation properties of the Discontinuous Galerkin method. There it can be found that one of the first studies presented about this subject is by Johnson and Pitkäranta [45]. In [45], Johnson and Pitkäranta included a Fourier analysis of the Discontinuous Galerkin method for the case $p=1$ and showed that the eigenvalue of the amplification matrix is accurate to order 4. Lowrie [57] performed a Fourier analysis of the space-time Discontinuous Galerkin algorithm, for $p$ up to $p=3$, for a one-dimensional scalar advection equation. He showed that the eigenvalue is accurate to order $2 p+2$ (locally) which results in a global order $2 p+1$ decay of the evolution component of the numerical error. In [43], Hu, Hussaini and Rasetarinera studied numerical dispersion and dissipation errors of the Discontinuous Galerkin method for both the one and two-dimensional wave equation. The same authors (this time in different order), recently showed numerically, that dispersion and dissipation errors decay at order $2 p+2$ when the exact characteristic splitting formula, for solving the one-dimensional Riemann problem at element interfaces, is used ([65]). As will be shown in this chapter, the analysis, for any order $p$, results in an eigenvalue problem, where at least one of the eigenvalues or modes accurately approximates a physical wave. Most of the other eigenvalues are spurious and
propagate unphysically. Always one of the eigenvalues is most accurate, in literature this mode is often called the physical wave. In [42], Hu and Atkins showed analytically, that the numerical dispersion relation, for the physically (most) accurate mode, is accurate to $h^{2 p+2}$. They furthermore show that, due to "mode decoupling", the numerical reflections, when they occur, are always in the form of the spurious non-physical mode.

In all of the mentioned publications the presented results are limited to a certain value of $p$. In Hagmeijer et al. [37], the governing characteristic polynomial is identified for any order $p$ for the semi-discrete algorithm obtained from applying the Discontinuous Galerkin method to the one-dimensional scalar advection equation, where the exact solution of the Riemann problem at element interfaces is used. The characteristic polynomial is of degree $p+1$ and has $p+1$ distinct roots, the eigenvalues. In this chapter the characteristic polynomial will be identified for the semi-discretization algorithm obtained from applying the Discontinuous Galerkin method to the 1D scalar advection equation employing the Lax-Friedrichs flux at element interfaces rather than the exact solution of the Riemann problem. It will be shown that the characteristic polynomial is identical regardless of the kind of polynomial basis used, as long as the basis function span the same approximate solution space. The $p+1$ roots of the polynomial are numerically investigated and results will be shown for $p$ up to $p=100$. Although the numerically obtained roots are of machine accuracy, it is noted that for $p=8$ or higher, this is not enough to accurately represent the (most accurate) physical mode. For a qualitative discussion this is not a problem at all, however, for a quantitative discussion, e.g. in defining error slopes, it is.

The characteristic polynomial is presented in the form of a theorem in section 4.3.2. It is preceded by a description of the Discontinuous Galerkin method applied to a model equation, the one-dimensional advection equation in section 4.2. It is furthermore shown how this equation is related to the one-dimensional linearized Euler equations. This relation shows why the one-dimensional advection equation is a good model problem to analyze.

Some of the difficulties encountered when describing the discontinuous Galerkin space discretization in three-dimensions, as in chapter 3, are less pronounced or even not present at all in one spatial dimension. When describing the Discontinuous Galerkin space discretization in this chapter, the notation might therefore differ from the one presented in chapter 3. Because of the great similarity with the description presented in chapter 3 the description of the Discontinuous Galerkin space discretization in this chapter will be somewhat more concise, however, still quite complete.

In section 4.2 the scalar advection is introduced, and it is shown how this equation is related to the linearized Euler equations in one-dimension. The main body of section 4.2 describes how applying the Discontinuous Galerkin method for the space discretization results in a system of equations. This system of equations is further discretized in time in section 6.29 by applying the Euler explicit, Euler implicit and the multi-stage Runge-Kutta time integration. In section 4.3.1 the dispersion relation of the semi-discrete system is identified, and is followed by the characteristic polynomial in section 4.3.2. Subsequently, in section 4.3.3, a numerical procedure is presented which can be used to obtain the $p+1$ roots of the characteristic polynomial. Finally results of analyzing the semi-discrete and fully discrete algorithm are presented.

### 4.2 Discontinuous Galerkin spatial discretization in 1D

Consider a domain $\Omega=[0,1]$ on which we want to solve the one-dimensional linear equation:

$$
\begin{equation*}
L(u(x, t))=0, \tag{4.1}
\end{equation*}
$$

where the linear operator $L$ is given by:

$$
\begin{equation*}
L \equiv \frac{\partial}{\partial t}+a \frac{\partial}{\partial x} . \quad a>0 \tag{4.2}
\end{equation*}
$$

The coefficient $a$ is assumed constant. Eq.(4.1) with the linear operator given by Eq.(4.2) is known as the one-dimensional scalar advection equation. Let $\mathcal{U}$ be the space of functions $u(x, t)$, where $x \in \Omega$ and $t \in I_{t}$. Let us define on $\Omega$ the space $\mathcal{V}=\left\{w(x) \in L^{2}(\Omega)\right\}$ of space-dependent functions and on $I_{t}$ the space $\mathcal{T}=\left\{\tau(t) \in L^{2}\left(I_{t}\right)\right\}$ of time-dependent functions, such that we may write for the solution space $\mathcal{U} \equiv \mathcal{V} \times \mathcal{T}$. In subsequent sections we will make extensive use of the inner product defined on the Hilbert space $L^{2}(\Omega)$ :

$$
\begin{equation*}
(f, g) \equiv \int_{\Omega} f(x) g(x) d x, \quad f, g \in L^{2}(\Omega) \tag{4.3}
\end{equation*}
$$

The one-dimensional scalar advection equation is related to the dimensionless linearized Euler equations in one spatial dimension:

$$
\frac{\partial \mathbf{u}}{\partial t}+A \frac{\partial \mathbf{u}}{\partial x}=0, \quad \mathbf{u}=\left(\begin{array}{c}
\rho  \tag{4.4}\\
u \\
p
\end{array}\right), \quad A=\left[\begin{array}{ccc}
M & 1 & 0 \\
0 & M & 1 \\
0 & 1 & M
\end{array}\right]
$$

The relation becomes apparent when realizing that the matrix $A$ has three linearly independent eigenvectors, corresponding to the eigenvalues $M+1, M, M-1$ of $A$, therefore $A$ can be diagonalized:

$$
A=R \Lambda R^{-1}, \quad R=\left[\begin{array}{ccc}
1 & 1 & 1  \tag{4.5}\\
1 & 0 & -1 \\
1 & 0 & 1
\end{array}\right], \quad \Lambda=\operatorname{diag}(M+1, M, M-1)
$$

Hence:

$$
\begin{equation*}
\frac{\partial \mathbf{u}}{\partial t}+A \frac{\partial \mathbf{u}}{\partial x}=\frac{\partial \mathbf{u}}{\partial \mathbf{w}} \frac{\partial \mathbf{w}}{\partial t}+A \frac{\partial \mathbf{u}}{\partial \mathbf{w}} \frac{\partial \mathbf{w}}{\partial x}=0, \quad \frac{\partial \mathbf{u}}{\partial \mathbf{w}}=R \tag{4.6}
\end{equation*}
$$

and

$$
\frac{\partial \mathbf{w}}{\partial t}+R^{-1} A R \frac{\partial \mathbf{w}}{\partial x}=\frac{\partial \mathbf{w}}{\partial t}+\Lambda \frac{\partial \mathbf{w}}{\partial x}=0, \quad \mathbf{w}=\left(\begin{array}{c}
\frac{1}{2}(p+u)  \tag{4.7}\\
\rho-p \\
\frac{1}{2}(p-u)
\end{array}\right)
$$

The components of $\mathbf{w}$ are the so-called Riemann invariants. The Riemann invariant $w_{i}$ is constant along $\frac{d w_{i}}{d t}=\Lambda_{i}$, i.e.

$$
\begin{align*}
& p+u=\text { constant along } \frac{d x}{d t}=M+1 \\
& p-\rho=\text { constant along } \frac{d x}{d t}=M  \tag{4.8}\\
& p-u=\text { constant along } \frac{d x}{d t}=M-1
\end{align*}
$$

The system of equations Eq.(4.7) is equivalent to the system Eq.(4.4). Eq.(4.7) represents a system of three uncoupled equations. Each of the three equations is of the form of the 1D scalar advection equation and can be analyzed separately. Hence, the results which will be obtained in this chapter for the 1D scalar advection equation can be applied to the linearized Euler equations in 1D as well.

Throughout the sections describing the Discontinuous Galerkin method, sections 4.2.1 and 4.2.2, we will adopt the Einstein summation convention, except for the index $j$ and except when stated otherwise.

### 4.2.1 Discontinuous Galerkin space discretization employing monomials

The semi-discrete discontinuous Galerkin approximation of Eq.(4.1) is obtained as follows: The domain $\Omega$ is partitioned into $N$ non-overlapping elements $\Omega_{j}$ :

$$
\begin{equation*}
\bar{\Omega}=\bigcup_{j=1}^{N} \bar{\Omega}_{j}, \quad \bar{\Omega}=\Omega+\partial \Omega . \tag{4.9}
\end{equation*}
$$

$\bar{\Omega}$ is called the closure of $\Omega$. The closure of element $j$ is given by $\bar{\Omega}_{j}=\Omega_{j}+\partial \Omega_{j}$, where $\partial \Omega_{j}=\{(j-1) \Delta x, j \Delta x\}$. In the current case we use a uniform grid and have $\Omega_{j}=$ $((j-1) \Delta x, j \Delta x)$ with $\Delta x \equiv N^{-1}$. Fig.(4.1) gives an illustration of the domain subdivision and the related element and interface numbering.


Figure 4.1: Element and interface numbering.

We consider an approximate solution $u_{h} \in \mathcal{U}_{h} \equiv \mathcal{V}_{h} \times \mathcal{T}$, where

$$
\begin{equation*}
\mathcal{V}_{h} \equiv \operatorname{span}\left\{b_{j k}\right\}, \quad j=0, \ldots, N, \quad k=0, \ldots, p \tag{4.10}
\end{equation*}
$$

Since the set $\left\{b_{j k}\right\}$ consists of linearly independent elements, $u_{h} \in \mathcal{V}_{h}$ has a unique representation:

$$
\begin{equation*}
u_{h}(x, t)=\sum_{j=1}^{N} v_{j k}(t) b_{j k}(x), \quad v_{j k}(t) \in \mathcal{T} \tag{4.11}
\end{equation*}
$$

where, as explained in section 4.2, we have implicit summation over $k$. So, within an element $\Omega_{j}$ we have:

$$
\begin{equation*}
u_{j}(x, t) \equiv v_{j k}(t) b_{j k}(x), \quad x \in \Omega_{j} \tag{4.12}
\end{equation*}
$$

The set $\left\{b_{j k}\right\}$ defines the space of polynomial functions of degree $\leq p$. We choose the set $\left\{b_{j k}\right\}$ as:

$$
b_{j k}(x)=\left\{\begin{array}{cl}
0, & x \notin \Omega_{j},  \tag{4.13}\\
\sqrt{\frac{2 k+1}{\Delta x}}\left(\xi^{j}(x)\right)^{k}, & x \in \Omega_{j},
\end{array}\right.
$$

where the map $\xi^{j}(x)$ is defined as:

$$
\begin{equation*}
\xi^{j}(x): \Omega_{j} \mapsto(-1,1), \quad \xi^{j}(x)=\frac{x-\left(j-\frac{1}{2}\right) \Delta x}{\frac{1}{2} \Delta x} \tag{4.14}
\end{equation*}
$$

Fig.(4.2) gives an example of the basis function for $k=0$ (top) and of the basis function for $k=1$ (bottom). In contrast to the 3D method, the basis functions have been normalized such


Figure 4.2: Top: representation of the basis function for $k=0$. bottom: representation of the basis function for $k=1$.
that $\left(b_{j k}, b_{j k}\right)=1$.
The Discontinuous Galerkin approximation of Eq.(4.1) on $\Omega$ is now formulated as follows: For every $t$, find $u_{h} \in \mathcal{V}_{h} \times \mathcal{T}$, where $u_{h}$ is given by Eq.(4.11), such that:

$$
\begin{equation*}
\left(L\left(u_{h}\right), b_{j m}\right)=0, \quad \forall m, j \tag{4.15}
\end{equation*}
$$

Here we introduced the short-hand notation $\forall m, j$, which implies $\forall m\{0, \ldots, p\}$ and $\forall j\{1, \ldots, N\}$, for convenience. Employing Eq.(4.1) we obtain from expanding Eq.(4.15):

$$
\begin{equation*}
\frac{d v_{j k}}{d t}\left(b_{j k}, b_{j m}\right)+\left(\frac{d f_{j}}{d x}, b_{j m}\right)=0, \quad \forall m, j \tag{4.16}
\end{equation*}
$$

where:

$$
\begin{equation*}
f_{j} \equiv a u_{j} \tag{4.17}
\end{equation*}
$$

is a linear flux. Partial integration of the second inner product in Eq.(4.16) results in:

$$
\begin{equation*}
\frac{d v_{j k}}{d t}\left(b_{j k}, b_{j m}\right)-\left(f_{j}, \frac{d b_{j m}}{d x}\right)+\left[b_{j m} f_{j}\right]_{(j-1) \Delta x}^{j \Delta x}=0, \quad \forall m, j \tag{4.18}
\end{equation*}
$$

From Eq.(4.13) we observe that for $x \in[(j-1) \Delta x, j \Delta x]$ the basis functions are only nonzero in the interior of element $j$. The points $x=(j-1) \Delta x$ and $x=j \Delta x$ are not elements of the interior $\left(\Omega_{j}\right)$ but elements of $\partial \Omega_{j}$ and the result of the last term on the left-hand-side of Eq.(4.18) is, strictly taken, zero. However, upon defining:

$$
\begin{align*}
b_{k}^{+} & \equiv \lim _{x \uparrow j \Delta x} b_{j k}(x)=\sqrt{\frac{2 k+1}{\Delta x}}  \tag{4.19}\\
b_{k}^{-} & \equiv \lim _{x \downarrow(j-1) \Delta x} b_{j k}(x)=(-1)^{k} \sqrt{\frac{2 k+1}{\Delta x}}, \tag{4.20}
\end{align*}
$$

the basis functions are extended to $\partial \Omega_{j}$.
To couple the system of equations represented by Eq.(4.18) we introduce, in analogy with the 3D description presented in chapter 3, the (local) Lax-Friedrichs flux as approximate Riemann solver:

$$
\begin{equation*}
f_{j}(j \Delta x, t) \approx \frac{1}{2}\left[f_{j+1}(j \Delta x, t)+f_{j}(j \Delta x, t)-\alpha \theta\left\{u_{j+1}(j \Delta x, t)-u_{j}(j \Delta x, t)\right\}\right] \tag{4.21}
\end{equation*}
$$

where, $\theta$ is an estimation of the largest eigenvalue of the Jacobian $\frac{\partial}{\partial u}\left(f\left(u_{h}(x, t)\right) n\right)$ for $(x, t)$ in a neighborhood of the interface between the elements, where $n$ is the outward pointing unit normal vector (which is in 1D simply given by: $n= \pm 1$ ). In the current, linear, case we have a uniform mean-flow and may therefore use $\theta=|a|=a$ :

$$
\begin{equation*}
f_{j}(j \Delta x, t) \approx a\left[\frac{1-\alpha}{2} b_{k}^{-} v_{j+1 k}(t)+\frac{1+\alpha}{2} b_{k}^{+} v_{j k}(t)\right] . \tag{4.22}
\end{equation*}
$$

The factor $\alpha \geq 1$ controls the upwinding. The Lax-Friedrichs flux formally requires $\alpha>$ 1 , however, we allow $\alpha=1$ as well because it provides the exact solution of the onedimensional Riemann problem at the interfaces as a special case.

The inner products of Eq.(4.18) are given by:

$$
\begin{align*}
\left(b_{j k}, b_{j m}\right) & =\frac{1}{2} \frac{\sqrt{2 k+1} \sqrt{2 m+1}}{k+m+1}\left[1-(-1)^{k+m+1}\right],  \tag{4.23}\\
\left(b_{j k}, \frac{d b_{j m}}{d x}\right) & = \begin{cases}0, & k=0, m=0 \\
\frac{1}{\Delta x} \frac{m}{k+m} \sqrt{2 k+1} \sqrt{2 m+1}\left[1-(-1)^{k+m}\right], & k+m>0\end{cases} \tag{4.24}
\end{align*}
$$

For $m=k$ Eq.(4.23) simplifies to:

$$
\begin{equation*}
\left(b_{j k}, b_{j k}\right)=1 \tag{4.25}
\end{equation*}
$$

We will conclude this section by writing out the semi-discrete algorithm, which is obtained upon substitution of Eq.(4.22) into Eq.(4.18):

$$
\begin{align*}
& \left(b_{j k}, b_{j m}\right) \frac{d v_{j k}}{d t}-\left(b_{j k}, \frac{d b_{j m}}{d x}\right) a v_{j k}+a\left[\frac{1-\alpha}{2} b_{k}^{-} b_{m}^{+} v_{j+1 k}+\frac{1+\alpha}{2} b_{k}^{+} b_{m}^{+} v_{j k}\right]- \\
& a\left[\frac{1-\alpha}{2} b_{k}^{-} b_{m}^{-} v_{j k}+\frac{1+\alpha}{2} b_{k}^{+} b_{m}^{-} v_{j-1 k}\right]=0, \quad \forall m, j \tag{4.26}
\end{align*}
$$

### 4.2.2 Alternative basis sets

Instead of using piecewise monomials for the basis set $\left\{b_{j k}\right\}$, other basis functions, such as piecewise Legendre polynomials, Hermite polynomials, Chebychev polynomials, etc., can be employed. In this section we investigate the relation between the semi-discretization employing piecewise monomial basis functions and piecewise Legendre polynomials, assuming both span the space $\mathcal{V}_{h}$. Relations between any other two basis sets spanning the same space $\mathcal{V}_{h}$, can be obtained in a similar way. In Hagmeijer et al. [37] the basis set is chosen as:

$$
\bar{b}_{j k}(x)= \begin{cases}0, & x \notin \Omega_{j}  \tag{4.27}\\ \sqrt{\frac{2 k+1}{\Delta x}} P_{k}\left(\xi_{j}(x)\right), & x \in \Omega_{j}\end{cases}
$$

where we use the notation $\bar{b}_{j k}$ to distinguish the basis functions of Eq.(4.13) from the basis functions given in Eq.(4.27). $P_{k}$ denotes the $k^{t h}$-Legendre polynomial and $\xi_{j}(x)$ is defined in Eq.(4.14). The basis functions $\left\{b_{j k}\right\}$ and $\left\{\bar{b}_{j k}\right\}$ both span the same subspace $\mathcal{V}_{h}$ :

$$
\begin{equation*}
\mathcal{V}_{h}=\operatorname{span}\left\{b_{j k}\right\}=\operatorname{span}\left\{\bar{b}_{j k}\right\} \tag{4.28}
\end{equation*}
$$

Since the set $\left\{b_{j k}\right\}$ consists of linearly independent elements, $u_{h} \in \mathcal{U}_{h}$ has a unique representation given by Eq.(4.11):

$$
u_{h}(x, t)=\sum_{j=1}^{N} v_{j k}(t) b_{j k}(x), \quad v_{j k}(t) \in \mathcal{T}
$$

However, the set $\left\{\bar{b}_{j k}\right\}$ also consists of linearly independent elements, and $u_{h} \in \mathcal{U}_{h}$ therefore also has the unique representation:

$$
\begin{equation*}
u_{h}(x, t)=\sum_{j=1}^{N} \bar{v}_{j k}(t) \bar{b}_{j k}(x), \quad \bar{v}_{j k}(t) \in \mathcal{T} . \tag{4.29}
\end{equation*}
$$

Each Legendre polynomial, $P_{k}(\xi)$, can be expressed as a linear combination of monomials, $\xi^{k}$, see for example [64]. Therefore the following relation between a monomial of the basis set $\left\{b_{j k}\right\}$ and Legendre polynomials of basis set $\left\{\bar{b}_{j k}\right\}$ exists:

$$
\begin{equation*}
b_{j k}=r_{k m} \bar{b}_{j m}, \tag{4.30}
\end{equation*}
$$

where the coefficients $r_{k m}$ are constant and independent of the index $j$. Substitution of Eq.(4.30) into Eq.(4.11) results in:

$$
\begin{equation*}
u_{h}(x, t)=\sum_{j=0}^{N} v_{j k}(t)\left(r_{k m} \bar{b}_{j m}(x)\right) \tag{4.31}
\end{equation*}
$$

and we obtain the relation:

$$
\begin{equation*}
\bar{v}_{j m}=v_{j k} r_{k m} \tag{4.32}
\end{equation*}
$$

The discontinuous Galerkin formulation presented in Eq.(4.15) can now be written in terms of Legendre polynomials:

$$
\begin{align*}
\left(L\left(u_{h}\right), b_{j m}\right) & =\left(L\left(\sum_{j=1}^{N} v_{j k} b_{j k}\right), b_{j m}\right)=\left(L\left(\sum_{j=1}^{N} \bar{v}_{j k} \bar{b}_{j k}\right), r_{m l} \bar{b}_{j l}\right) \\
& =r_{m l}\left(L\left(\sum_{j=1}^{N} \bar{v}_{j k} \bar{b}_{j k}\right), \bar{b}_{j l}\right) \tag{4.33}
\end{align*}
$$

In Hagmeijer et al. [37] the weak formulation:

$$
\begin{equation*}
\left(L\left(u_{h}\right), \bar{b}_{j m}\right)=\left(L\left(\sum_{j=1}^{N} \bar{v}_{j k} \bar{b}_{j k}\right), \bar{b}_{j m}\right)=0 \tag{4.34}
\end{equation*}
$$

has been evaluated. When employing Legendre polynomials the inner products of Eqs.(4.23) and (4.24) become:

$$
\begin{align*}
\left(\bar{b}_{j k}, \bar{b}_{j m}\right) & =\delta_{k m},  \tag{4.35}\\
\left(\bar{b}_{j k}, \frac{d \bar{b}_{j m}}{d x}\right) & =\left\{\begin{array}{lr}
\frac{2}{\Delta x} \sqrt{2 k+1} \sqrt{2 m+1}, & m>k, \quad m-k \quad \text { odd }, \\
0, & \text { otherwise } .
\end{array}\right. \tag{4.36}
\end{align*}
$$

Eq.(4.35) reflects that the set $\left\{\bar{b}_{j k}\right\}$ is orthonormal. For the basis functions on the interfaces we have:

$$
\begin{align*}
& \bar{b}_{k}^{+} \equiv \lim _{x \uparrow j \Delta x} \bar{b}_{j k}(x)=\sqrt{\frac{2 k+1}{\Delta x}},  \tag{4.37}\\
& \bar{b}_{k}^{-} \equiv \lim _{x \downarrow(j-1) \Delta x} \bar{b}_{j k}(x)=(-1)^{k} \sqrt{\frac{2 k+1}{\Delta x}} . \tag{4.38}
\end{align*}
$$

From which it is noted that the $\bar{b}_{k}^{+}=b_{k}^{+}$and $\bar{b}_{k}^{-}=b_{k}^{-}$.
In the beginning of this section it was mentioned that other basis functions, which are defined such that they span $\mathcal{V}_{h}$, e.g. Hermite polynomials or Chebychev polynomials, etc., could be used in the formulation. These polynomials can also be expressed in terms of the monomial basis functions $b_{j k}$, and therefore we can obtain similar expressions as Eq.(4.30) for these basis functions as well. This has important implications as we will see later.

In the next section both the algorithm employing monomial basis functions and the algorithm employing Legendre polynomials as basis functions are considered in vector notation. In section 4.3 it will be shown that the two algorithms have equivalent wave-propagation properties (when both basis sets span the same approximation space). The actual wavepropagation analysis is then conducted, conveniently, for the algorithm employing Legendre polynomials as basis functions.

### 4.2.3 Summary in vector notation

In section 4.3 the analysis of the semi-discrete algorithm, as presented above in Eq.(4.26), will be conducted. In the analysis we will distinguish between the case in which we employ the exact flux $(\alpha=1)$ and the case in which we employ the Lax-Friedrichs flux $(\alpha>1)$. These two cases can be presented more distinctively, upon introducing:

$$
\begin{equation*}
\epsilon \equiv \frac{\alpha-1}{2} . \tag{4.39}
\end{equation*}
$$

Then $\epsilon=0$ gives us the algorithm employing the exact flux and $\epsilon>0$ gives us the algorithm employing the Lax-Friedrichs flux. The analysis is most conveniently performed when the summations over $k$ for all $m$ in Eq.(4.26) is worked out and the resulting sets of equations are written in vector notation. Define:

$$
\begin{equation*}
\mathbf{v}_{j} \in \mathbb{R}^{p+1}, \quad \mathbf{v}_{j} \equiv\left(v_{j 0}, v_{j 1}, \ldots, v_{j p}\right)^{T} \tag{4.40}
\end{equation*}
$$

and, upon introducing:

$$
\begin{equation*}
d_{m k} \equiv \sqrt{2 m+1} \sqrt{2 k+1} \tag{4.41}
\end{equation*}
$$

define the so-called mass matrix $M$ and the matrices $A, B$ and $C$ :

$$
\begin{align*}
M_{m k} & \equiv\left(b_{j m}, b_{j k}\right)=\frac{d_{m k}}{2(m+k+1)}\left[1-(-1)^{m+k+1}\right]  \tag{4.42}\\
\frac{1}{\Delta x} A_{m k} & \equiv b_{m}^{+} b_{k}^{+}-\left(b_{j k}, \frac{\partial b_{j m}}{\partial x}\right) \\
& =\frac{d_{m k}}{\Delta x}\left\{\begin{array}{ll}
1, & k=0, m=0 \\
1-\frac{m}{k+m}\left[1-(-1)^{k+m}\right],
\end{array} \quad \text { otherwise },\right.  \tag{4.43}\\
\frac{1}{\Delta x} B_{m k} & \equiv b_{m}^{-} b_{k}^{+}=\frac{d_{m k}}{\Delta x}(-1)^{m},  \tag{4.44}\\
\frac{1}{\Delta x} C_{m k} & \equiv b_{m}^{+} b_{k}^{+}+b_{m}^{-} b_{k}^{-}=\frac{d_{m k}}{\Delta x}\left[1+(-1)^{k+m}\right] \tag{4.45}
\end{align*}
$$

Eq.(4.26) can now be written as:

$$
\begin{equation*}
M \frac{d \mathbf{v}_{j}}{d t}+\frac{a}{\Delta x}\left[A \mathbf{v}_{j}-B \mathbf{v}_{j-1}-\epsilon\left[B \mathbf{v}_{j-1}-C \mathbf{v}_{j}+B^{T} \mathbf{v}_{j+1}\right]\right]=\mathbf{0} \tag{4.46}
\end{equation*}
$$

In addition, let us introduce the following operators ([48]):

$$
\begin{array}{ll}
I & : I \mathbf{v}_{j}=\mathbf{v}_{j} \\
S_{+} & :  \tag{4.47}\\
S_{+} \mathbf{v}_{j}=\mathbf{v}_{j+1}, \\
S_{-} & : \\
S_{-} \mathbf{v}_{j}=\mathbf{v}_{j-1}
\end{array}
$$

then we can define a general operator $L_{s}$ as:

$$
\begin{equation*}
L_{s}=\frac{d}{d t}-\frac{a}{\Delta x} K\left(S_{-}, S_{+}\right) \tag{4.48}
\end{equation*}
$$

where $K\left(S_{-}, S_{+}\right)$is a polynomial in $S_{-}$and $S_{+}$with coefficients consisting of $(p+1) \times$ $(p+1)$-matrices. With this notation, the system of equations, Eq.(4.46), for the monomial coefficients, $\mathbf{v}_{j}$, in Eq.(4.11), can conveniently be written as:

$$
\begin{equation*}
L_{s}\left(\mathbf{v}_{j}\right)=\frac{d \mathbf{v}_{j}}{d t}-\frac{a}{\Delta x} K \mathbf{v}_{j}=\mathbf{0} \tag{4.49}
\end{equation*}
$$

where

$$
\begin{equation*}
-K=M^{-1}\left\{A-B S_{-}-\epsilon\left[B S_{-}-C+B^{T} S_{+}\right]\right\} \tag{4.50}
\end{equation*}
$$

Similarly, the system of equations for the Legendre polynomial coefficients, $\overline{\mathbf{v}}_{j}$, in Eq.(4.29), can be derived from Eq.(4.34):

$$
\begin{equation*}
L_{s}\left(\overline{\mathbf{v}}_{j}\right)=\frac{d \overline{\mathbf{v}}_{j}}{d t}-\bar{K} \overline{\mathbf{v}}_{j} \tag{4.51}
\end{equation*}
$$

where

$$
\begin{equation*}
-\bar{K}=\bar{A}-\bar{B} S_{-}-\epsilon\left[\bar{B} S_{-}-\bar{C}+\bar{B}^{T} S_{+}\right] \tag{4.52}
\end{equation*}
$$

and where the matrices $\bar{A}, \bar{B}$ and $\bar{C}$ are given by:

$$
\begin{align*}
\frac{1}{\Delta x} \bar{A}_{m k} & \equiv \bar{b}_{m}^{+} \bar{b}_{k}^{+}-\left(\bar{b}_{j k}, \frac{\partial \bar{b}_{j m}}{\partial x}\right)=\frac{d_{m k}}{\Delta x}\left[1-2 \phi_{m k}\right]  \tag{4.53}\\
\frac{1}{\Delta x} \bar{B}_{m k} & \equiv \bar{b}_{m}^{-} \bar{b}_{k}^{+}=\frac{d_{m k}}{\Delta x}(-1)^{m}  \tag{4.54}\\
\frac{1}{\Delta x} \bar{C}_{m k} & \equiv \bar{b}_{m}^{+} \bar{b}_{k}^{+}+\bar{b}_{m}^{-} \bar{b}_{k}^{-}=\frac{d_{m k}}{\Delta x}\left[1+(-1)^{m+k}\right] \tag{4.55}
\end{align*}
$$

In Eq.(4.53) we introduced:

$$
\phi_{m k} \equiv\left\{\begin{array}{lcc}
1, & m>k, & m-k  \tag{4.56}\\
0, & \text { odd } \\
0, & \text { else }
\end{array}\right.
$$

Legendre polynomials have the advantage of being orthogonal, leading to a "mass matrix" which is given by the identity matrix, $\left(\bar{b}_{j k}, \bar{b}_{j m}\right)=\delta_{k m}$, which simplifies the system of equations, see also Eqs.(4.49) and (4.51). From Eqs.(4.54) and (4.44) it is observed that $\bar{B}=B$. In addition it is observed from Eqs.(4.55) and (4.45) that $\bar{C}=C$. The two systems of Eq.(4.49) and Eq.(4.51) are related, as is shown by Eq.(4.33). Writing Eq.(4.33) in the form of Eq.(4.30) and Eq.(4.32) and with $R=\left[r_{m k}\right]$ we obtain:

$$
\begin{equation*}
M \frac{d \mathbf{v}_{j}}{d t}-\frac{a}{\Delta x} E \mathbf{v}_{j}=R\left[\frac{d \overline{\mathbf{v}}_{j}}{d t}-\frac{a}{\Delta x} \bar{K} \overline{\mathbf{v}}_{j}\right]=\mathbf{0} \tag{4.57}
\end{equation*}
$$

where

$$
\begin{equation*}
E \equiv M K \tag{4.58}
\end{equation*}
$$

then we obtain upon employing Eq.(4.32), from which it follows that $\overline{\mathbf{v}}_{j}=R^{T} \mathbf{v}_{j}$, the relation:

$$
\begin{equation*}
M \frac{d \mathbf{v}_{j}}{d t}-\frac{a}{\Delta x} E \mathbf{v}_{j}=R R^{T} \frac{d \mathbf{v}_{j}}{d t}-\frac{a}{\Delta x} R \bar{K} R^{T} \mathbf{v}_{j}=\mathbf{0} \tag{4.59}
\end{equation*}
$$

From which it is observed that:

$$
\begin{align*}
& M=R R^{T}  \tag{4.60}\\
& E=R \bar{K} R^{T} \tag{4.61}
\end{align*}
$$

In addition, it was found that:

$$
\begin{align*}
& A=R \bar{A} R^{T}  \tag{4.63}\\
& B=R \bar{B} R^{T}  \tag{4.64}\\
& C=R \bar{C} R^{T} \tag{4.65}
\end{align*}
$$

In other words, pre-multiplying by $R^{-1}$ and post-multiplying by $R^{-T}$, transforms Eq.(4.49) into Eq.(4.51). Note that $\operatorname{det}(R) \neq 0$. Similar relations can be obtained between monomial basis functions and other basis functions which span $\mathcal{V}_{h}$, such as Hermite polynomials or Chebychev polynomials. For each of these relations a different matrix $R$ will be obtained, however Eq.(4.59) will be applicable in all of these cases.

### 4.2.4 Time integration

In the preceding sections the space-discretization of the scalar advection equation, Eq.(4.1), employing the discontinuous Galerkin method with monomial basis functions and with Legendre polynomials as basis functions have been presented, to result in two semi-discrete algorithms. Since the semi-discretization algorithm based on monomials can be transformed into the algorithm based on Legendre polynomials, it suffices to consider only the time integration in combination with the space discretization based on Legendre polynomials. In the next section it will be shown that both algorithms have equivalent wave-propagation properties. The actual wave-propagation analysis will be conducted for the semi-discrete algorithm employing Legendre polynomials as basis functions.

In this section the semi-discrete algorithm will be discretized in time, to result in a fully discretized algorithm or scheme. For the time integration the Euler explicit, Euler implicit and multi-stage (low storage) Runge-Kutta time integration algorithms are considered.

Let us introduce the operator:

$$
\begin{equation*}
S^{+} \quad: \quad S^{+} \overline{\mathbf{v}}_{j}^{n}=\overline{\mathbf{v}}_{j}^{n+1}, \quad n \in \mathbb{N}^{+} \tag{4.66}
\end{equation*}
$$

where $\overline{\mathbf{v}}_{j}^{n}$ and $\overline{\mathbf{v}}_{j}^{n+1}$ denote the solution at time $n \Delta t$ and $(n+1) \Delta t$, respectively, where $\Delta t>0$ is a sufficiently small time increment. In this section the operators for the fully discretized algorithm will be presented in the following form:

$$
\begin{equation*}
L_{f}=P\left(S_{-}, S_{+}\right) S^{+}-Q\left(S_{-}, S_{+}\right), \tag{4.67}
\end{equation*}
$$

where $P$ and $Q$ differ for the different fully discretized algorithms. The operators $P$ and $Q$ can be expressed in terms of $\bar{K}$, which has been introduced in the preceding section. The fully discrete algorithms can then be written in the form:

$$
\begin{equation*}
L_{f}\left(\overline{\mathbf{v}}_{j}^{n}\right)=\mathbf{0} \tag{4.68}
\end{equation*}
$$

Let us furthermore define:

$$
\begin{equation*}
\sigma \equiv \frac{a \Delta t}{\Delta x} \tag{4.69}
\end{equation*}
$$

which is known as the CFL-number (after Courant, Friedrichs and Lewy).

## Euler explicit time integration

Upon applying the well-known Euler explicit time integration, see for example [40], to Eq.(4.51) we obtain:

$$
\begin{equation*}
\frac{\overline{\mathbf{v}}_{j}^{n+1}-\overline{\mathbf{v}}_{j}^{n}}{\Delta t}-\frac{a}{\Delta x} \bar{K} \overline{\mathbf{v}}_{j}^{n}=\mathbf{0} \tag{4.70}
\end{equation*}
$$

from which it is obtained:

$$
\left.\begin{array}{l}
P=I  \tag{4.71}\\
Q=I+\sigma \bar{K}
\end{array}\right\} \quad \Rightarrow \quad L_{f}\left(\overline{\mathbf{v}}_{j}^{n}\right)=\overline{\mathbf{v}}_{j}^{n+1}-[I+\sigma \bar{K}] \overline{\mathbf{v}}_{j}^{n}=\mathbf{0}
$$

This, simplest scheme, is an explicit scheme, since each discretized equation contains only one unknown (vector) at time level $n+1$. Although very simple, the Euler explicit time integration scheme is not known for its accuracy. It is furthermore known to lead to unstable schemes in many cases.

## Euler implicit time integration

From applying the Euler implicit time integration, see for example [40], to Eq.(4.51), one obtains:

$$
\begin{equation*}
\frac{\overline{\mathbf{v}}_{j}^{n+1}-\overline{\mathbf{v}}_{j}^{n}}{\Delta t}-\frac{a}{\Delta x} \bar{K} \overline{\mathbf{v}}_{j}^{n+1}=\mathbf{0} \tag{4.72}
\end{equation*}
$$

which, written in the form of Eq.(4.68), gives:

$$
\left.\begin{array}{l}
P=I-\sigma \bar{K}  \tag{4.73}\\
Q=I
\end{array}\right\} \quad \Rightarrow \quad L_{f}\left(\overline{\mathbf{v}}_{j}^{n}\right)=[I-\sigma \bar{K}] \overline{\mathbf{v}}_{j}^{n+1}-\overline{\mathbf{v}}_{j}^{n}=\mathbf{0} .
$$

## Runge-Kutta time integration, $N$-stage

The low-storage $N$-stage Runge-Kutta time integration can be presented by the following sequence, see for example [10], [40] or [44]:

$$
\begin{align*}
\overline{\mathbf{v}}_{j}^{n, 0} & =\overline{\mathbf{v}}_{j}^{n} \\
\overline{\mathbf{v}}_{j}^{n, k} & =\overline{\mathbf{v}}_{j}^{n}+\gamma_{k} \sigma_{k} \bar{K} \overline{\mathbf{v}}_{j}^{n, k-1}, \quad k=1, \ldots, N \\
\overline{\mathbf{v}}_{j}^{n+1} & =\overline{\mathbf{v}}_{j}^{n, N} \tag{4.74}
\end{align*}
$$

from which it can be derived:

$$
\begin{align*}
& P=I  \tag{4.75}\\
& Q=I+\gamma_{N} \sigma \bar{K}+\gamma_{N} \gamma_{N-1} \sigma^{2} \bar{K}^{2}+\ldots+\gamma_{N} \cdots \gamma_{1} \sigma^{N} \bar{K}^{N} \tag{4.76}
\end{align*}
$$

The operator for the fully discrete algorithm is now given by:

$$
\begin{equation*}
L_{f}\left(\overline{\mathbf{v}}_{j}^{n}\right)=\overline{\mathbf{v}}_{j}^{n+1}-\left[I+\gamma_{N} \sigma \bar{K}+\ldots+\gamma_{N} \cdots \gamma_{1} \sigma^{N} \bar{K}^{N}\right] \overline{\mathbf{v}}_{j}^{n}=\mathbf{0} . \tag{4.77}
\end{equation*}
$$

The classical four-stage Runge-Kutta scheme has coefficients $\gamma_{4}=1, \gamma_{3}=\frac{1}{2}, \gamma_{2}=\frac{1}{3}$, $\gamma_{1}=\frac{1}{4}$.

### 4.3 Analysis of semi-dicrete scheme

In this section the actual dissipation and dispersion error analysis of the semi-discretization of the one-dimensional scalar advection equation Eq.(4.1) is presented. The analysis will be performed assuming that the domain $\Omega$, presented in section 4.2.1, is now given by $\Omega=\mathbb{R}$.

First it will be shown that the two semi-discrete algorithms, the one obtained from employing monomial basis functions and the one obtained from employing Legendre polynomials as basis functions, have equivalent wave-propagation properties (when both basis sets span the same approximation space). The actual wave-propagation analysis is then conducted, conveniently, for the semi-discrete algorithm employing Legendre polynomials as basis functions.

### 4.3.1 Dispersion relation of semi-discrete system

In view of the fact that any wave can be written in terms of a Fourier integral, it is convenient to analyze the wave propagation performance of the (semi-) discretized scheme by considering a monochromatic wave of the form:

$$
\begin{equation*}
u(x, t) \sim e^{i(k x-\omega t)} \tag{4.78}
\end{equation*}
$$

which represents a sinusoidal wave with wavenumber $k$, frequency $\omega$ and where $i=\sqrt{-1}$. It is easily verified that the monochromatic wave, Eq.(4.78), satisfies the one-dimensional scalar advection equation, Eq.(4.1), when

$$
\begin{equation*}
k=\frac{\omega}{a} . \tag{4.79}
\end{equation*}
$$

Eq.(4.79) is called the dispersion relation of the exact equation. It relates the wavelength in space $\left(\lambda_{s}=\frac{2 \pi}{k}\right)$ to the wavelength in time $\left(\lambda_{t}=\frac{2 \pi}{\omega}\right)$ via

$$
\begin{equation*}
\frac{\lambda_{s}}{\lambda_{t}}=\frac{\omega}{k}=a . \tag{4.80}
\end{equation*}
$$

Any grid function can be represented by a sum over discrete monochromatic waves given by:

$$
\begin{equation*}
\sim e^{i\left(k x_{j}-\omega t\right)}, \quad x_{j}=\left(j-\frac{1}{2}\right) \Delta x \tag{4.81}
\end{equation*}
$$

In order to analyze the scheme or system, based on monomials, we substitute:

$$
\begin{equation*}
\mathbf{v}_{j}(t)=\hat{\mathbf{v}}(t) e^{i(j k \Delta x-\omega t)} \tag{4.82}
\end{equation*}
$$

into Eq.(4.49) and attempt to derive a condition to be satisfied by $\omega$ and $k$ to make Eq.(4.82) a solution of the system Eq.(4.49). Before we do that, let us substitute Eq.(4.82) into Eq.(4.47), to obtain: for the operators $S_{-}$and $S_{+}$:

$$
\begin{align*}
& S_{+}: \quad S_{+} \hat{\mathbf{v}} e^{i(j k \Delta x-\omega t)}=e^{i k \Delta x} \hat{\mathbf{v}} e^{i(j k \Delta x-\omega t)} \\
& S_{-}: \tag{4.83}
\end{align*} S_{-} \hat{\mathbf{v}} e^{i(j k \Delta x-\omega t)}=e^{-i k \Delta x} \hat{\mathbf{v}} e^{i(j k \Delta x-\omega t)} .
$$

from which we obtain for the scheme based on monomials:

$$
\begin{align*}
& K\left(S_{-}, S_{+}\right) \hat{\mathbf{v}} e^{i(j k \Delta x-\omega t)}=K\left(e^{-i k \Delta x}, e^{i k \Delta x}\right) \hat{\mathbf{v}} e^{i(j k \Delta x-\omega t)}  \tag{4.84}\\
& Q\left(S_{-}, S_{+}\right) \hat{\mathbf{v}} e^{i(j k \Delta x-\omega t)}=Q\left(e^{-i k \Delta x}, e^{i k \Delta x}\right) \hat{\mathbf{v}} e^{i(j k \Delta x-\omega t)} \tag{4.85}
\end{align*}
$$

With this in mind, we obtain upon substituting Eq.(4.82) into Eq.(4.49):

$$
\begin{equation*}
\left[-i \omega I-\frac{a}{\Delta x} K\left(e^{-i k \Delta x}, e^{i k \Delta x}\right)\right] \hat{\mathbf{v}} e^{i(j k \Delta x-\omega t)}=\mathbf{0} \tag{4.86}
\end{equation*}
$$

A non-trivial solution of Eq.(4.86) can be obtained only if:

$$
\begin{equation*}
\operatorname{det}(-K-\lambda I)=0, \quad \lambda \equiv \frac{i \omega \Delta x}{a} \tag{4.87}
\end{equation*}
$$

which is a necessary condition for $\omega, \Delta x$ and $a$, which must be satisfied in order to make $\mathbf{v}_{j}$ in Eq.(4.82) a possible solution of Eq.(4.49). Clearly $\lambda$ is an eigenvalue of the matrix $-K$. Eq.(4.87) is referred to as the dispersion relation of the discretized system Eq.(4.49). It results in $p+1$ eigenvalues $\lambda_{m}, m \in[1, p+1]$. The eigenvalues represent numerical approximations of $i \omega$, which can be seen from Eq.(4.87), i.e. $\frac{a}{\Delta x} \lambda_{m} \approx i \omega$. The eigenvalues $\lambda_{m}$ are in literature also called modes or roots. With these eigenvalues the numerical solution in an element $\Omega_{j}$ can be written as:

$$
\begin{equation*}
u_{h}(x, t)=e^{i j k \Delta x} \sum_{m=1}^{p+1} w_{m} e^{-\lambda_{m} \frac{a t}{\Delta x}}, \quad x \in \Omega_{j} \tag{4.88}
\end{equation*}
$$

where $w_{m}=w_{l m} b_{l}$ is the $m^{t h}$-eigenvector, corresponding to $\lambda_{m}$. The eigenvalues $\lambda_{m}$ are in general complex, with, as we will see in section $4.3 .4, \operatorname{Re}\left(\lambda_{m}\right) \leq 0$. Only the eigenvalues or modes that closely approximate the exact dispersion relation and which are approximately not dissipated numerically, will propagate like waves which are a solution of Eq.(4.1). We will call these specific modes the physical modes or principle roots, all other eigenvalues or modes will be called spurious or parasite modes of the numerical scheme. There is always at least one principle root or physical mode. With increasing polynomial degree $p$ more modes may propagate as physical modes.

In the remainder of this section we will adopt the short-hand notation:

$$
\begin{equation*}
\mu \equiv e^{i k \Delta x} \tag{4.89}
\end{equation*}
$$

from which one obtains from Eqs.(4.50):

$$
\begin{equation*}
-K=M^{-1}\left\{A-\mu^{-1} B-\epsilon\left[\mu^{-1} B-C+\mu B^{T}\right]\right\} \tag{4.90}
\end{equation*}
$$

The solutions $\lambda_{m}$, of Eq.(4.87), are functions of $\mu$ and depend on the parameter $\epsilon: \lambda_{m}=$ $\lambda_{m}(\mu ; \epsilon)$. For example, when $p=0$ it can be shown that the only eigenvalue is:

$$
\begin{equation*}
\lambda(\mu ; \epsilon)=1-\mu^{-1}+\epsilon\left[2-\left(\mu+\mu^{-1}\right)\right] . \tag{4.91}
\end{equation*}
$$

Hence, for the exact flux $(\epsilon=0)$ we have:

$$
\begin{equation*}
\lambda(\mu ; 0)=1-\mu^{-1} \tag{4.92}
\end{equation*}
$$

For $p=1$ the characteristic polynomial is given by:

$$
\begin{equation*}
\lambda^{2}+2\left[2-\mu^{-1}-\epsilon\left(4+\mu+\mu^{-1}\right)\right] \lambda+6\left[1+\mu^{-1}+\epsilon\left(\mu+\mu^{-1}\right)\right]=0 \tag{4.93}
\end{equation*}
$$

which leads to the following roots for $\epsilon=0$ :

$$
\begin{equation*}
\lambda_{1,2}(\mu ; 0)=2+\mu^{-1} \pm \sqrt{\left(2+\mu^{-1}\right)^{2}-6\left(1-\mu^{-1}\right)} \tag{4.94}
\end{equation*}
$$

For $\epsilon>0$ the expression becomes rather lengthy.
Analysis of the dispersion relation Eq.(4.87) is straightforward for $p \leq 1$ but for larger values of $p$ we have to determine roots of the $(p+1)^{t h}$-degree characteristic polynomial,
which is not trivial. However, before we start tackling this problem, it is noted that the analysis, which must lead to a characteristic polynomial, can also be performed for the algorithm obtained from employing (for example) Legendre polynomials as basis functions, Eq.(4.51). From substitution of $\overline{\mathbf{v}}_{j}=\hat{\overline{\mathbf{v}}} e^{i\left(k x_{j}-\omega t\right)}$ (Eq.(4.82)) into Eq.(4.51) we obtain the eigenvalue problem:

$$
\begin{equation*}
\operatorname{det}(-\bar{K}-\lambda I)=0 \tag{4.95}
\end{equation*}
$$

where

$$
\begin{equation*}
-\bar{K}=\bar{A}-\mu^{-1} \bar{B}-\epsilon\left[\mu^{-1} \bar{B}-\bar{C}+\mu \bar{B}^{T}\right] . \tag{4.96}
\end{equation*}
$$

In Eq.(4.59) both systems, i.e. the system based on monomials and the system based on Legendre polynomials, have been related. The same equation can also be used to relate the two eigenvalue problems. Substitution of Eq.(4.82) into Eq.(4.59) results in :

$$
\begin{equation*}
[-E-\lambda M] \mathbf{v}_{j}=R[-\bar{K}-\lambda I] R^{T} \mathbf{v}_{j}=\mathbf{0} \tag{4.97}
\end{equation*}
$$

from which we obtain:

$$
\begin{align*}
\operatorname{det}(-E-\lambda M) & =\operatorname{det}\left(R[-\bar{K}-\lambda I] R^{T}\right) \\
& =\operatorname{det}(R) \operatorname{det}(-\bar{K}-\lambda I) \operatorname{det}\left(R^{T}\right)=0 \tag{4.98}
\end{align*}
$$

Since the entries of the matrix $R$ are independent of $\lambda$ and $\mu$, and since $\operatorname{det}(R) \neq 0$, it is obtained from Eq.(4.98):

$$
\begin{equation*}
\operatorname{det}(-E-\lambda M)=0 \quad \Longleftrightarrow \quad \operatorname{det}(-\bar{K}-\lambda I)=0 \tag{4.99}
\end{equation*}
$$

Hence, it is concluded that the eigenvalues, obtained for the algorithm based on Legendre polynomials, are equivalent to the eigenvalues, obtained for the algorithm based on monomials. More general, it can be shown, that the eigenvalues obtained for algorithms based on any two sets of basis functions spanning $\mathcal{V}_{h}$, are identical. Therefore, instead of conducting the wave-propagation analysis of the algorithm based on monomials we may analyze the eigenvalues of the algorithm based on Legendre polynomials. In Hagmeijer et al. [37] the algorithm based on Legendre polynomials has been analyzed for $\epsilon=0$.

### 4.3.2 Characteristic polynomial for arbitrary values of $p$

In the current section the eigenvalues $\lambda$ of the matrix $-\bar{K}$, Eq.(4.99):

$$
\operatorname{det}(-\bar{K}-\lambda I)=0, \quad-\bar{K}=\bar{A}-\bar{B} S_{-}-\epsilon\left[\bar{B} S_{-}-\bar{C}+\bar{B}^{T} S_{+}\right]
$$

are considered. For values of $p$ larger than 1 , we have to determine the roots of the $(p+$ $1)^{t h}$-order characteristic complex polynomial, which is not at all straightforward. In addition, for large values of $p$ identification of the characteristic polynomial itself becomes non-trivial, since the calculation of the determinant of a $(p+1) \times(p+1)$-matrix involves $(p+1)$ ! terms when straight forward cofactor expansion is employed ([86]). However, the characteristic polynomial can simply be determined by the following theorem, which is an extension of the theorem presented in [37]:

Theorem 4.1 (Characteristic polynomial) Let, for some positive integer $p$, three $(p+1) \times$ ( $p+1$ )-matrices $\bar{A}, \bar{B}$ and $\bar{C}$ be defined as:

$$
\begin{align*}
\bar{A}_{m k} & \equiv\left\{\begin{array}{rc}
-d_{m k}, & m>k, m-k \text { odd }, \\
d_{m k}, & \text { otherwise },
\end{array}\right.  \tag{4.100}\\
\bar{B}_{m k} & \equiv\left\{\begin{array}{rc}
-d_{m k}, & \text { modd }, \\
d_{m k}, & \text { otherwise },
\end{array}\right.  \tag{4.101}\\
\bar{C}_{m k} & \equiv\left\{\begin{array}{rc}
2 d_{m k}, & m+k \text { even }, \\
0, & \text { otherwise },
\end{array}\right.  \tag{4.102}\\
& d_{m k}=\sqrt{2 m+1} \sqrt{2 k+1} \tag{4.103}
\end{align*}
$$

with $m, k \in[0, p]$. Then for $\mu \in \mathbb{C}, \mu \neq 1$, the eigenvalues $\lambda_{n}, n=0,1, \ldots, p+1$, of the matrix:

$$
\begin{equation*}
\bar{A}-\mu^{-1} \bar{B}-\epsilon\left[\mu^{-1} \bar{B}-\bar{C}+\mu \bar{B}^{T}\right] \tag{4.104}
\end{equation*}
$$

satisfy:

$$
\begin{equation*}
\left[(\mu-1) \phi_{0}^{p}\left(\lambda_{n}, \mu ; \epsilon\right)+(\mu+1) \phi_{1}^{p}\left(\lambda_{n}, \mu ; \epsilon\right)-\mu\right] \lambda_{n}^{p+1}=0 \tag{4.105}
\end{equation*}
$$

with

$$
\begin{align*}
\phi_{k}^{p}\left(\lambda_{n}, \mu ; \epsilon\right)= & \phi_{k+2}^{p}\left(\lambda_{n}, \mu ; \epsilon\right)-\frac{2 k+1}{\lambda_{n}}\left[2 \phi_{k+1}^{p}\left(\lambda_{n}, \mu ; \epsilon\right)-1-\epsilon\left(1-(-1)^{k} \mu\right)\right], \\
& k \in[0, p], \\
\phi_{k}^{p}\left(\lambda_{n}, \mu ; \epsilon\right)= & 0, \quad k>p . \tag{4.106}
\end{align*}
$$

Proof: The proof is given in appendix B.
For $\epsilon=0$, the theorem reduces to the theorem given in Hagmeijer et al. [37].

### 4.3.3 Numerical solution of characteristic polynomial

Now that we have obtained the characteristic polynomial of degree $p+1$, we are left with finding the $p+1$ roots of the complex polynomial. In this section a procedure for obtaining the roots is presented. The procedure was derived for the case $\epsilon=0$, and was presented in [37]. The procedure is furthermore similar to the procedure used by Hu and Atkins in [42].

In this section we consider roots $\lambda$ of Eq.(4.105), where Eq.(4.105) is written as:

$$
\begin{equation*}
f_{p}(\lambda, \mu ; \epsilon)=0 \tag{4.107}
\end{equation*}
$$

which is a polynomial of $\lambda$ of degree $p+1$ and furthermore a function of $\mu$. We are looking for the set $\Lambda(p, \epsilon)$ of roots $\lambda$, defined as:

$$
\begin{equation*}
\Lambda(p, \epsilon) \equiv\left\{\lambda \in \mathbb{C}\left|f_{p}(\lambda, \mu ; \epsilon)=0, \quad \mu \in \mathbb{C},|\mu|=1\right\}\right. \tag{4.108}
\end{equation*}
$$

Next we will present the solution procedure for the special case $\epsilon=0$, because it is more easily explained for this case than for the general case $\epsilon>0$. For $\epsilon>0$ the same procedure is used together with a few necessary modifications.

## Special case $\boldsymbol{\epsilon}=0$

We are looking for the set $\Lambda(p, 0)$. Next, define

$$
\begin{equation*}
\nu(\lambda) \equiv \frac{\phi_{1}^{p}(\lambda ; 0)-\phi_{0}^{p}(\lambda ; 0)}{1+\phi_{0}^{p}(\lambda ; 0)-\phi_{1}^{p}(\lambda ; 0)}, \tag{4.109}
\end{equation*}
$$

which satisfies:

$$
\begin{equation*}
f_{p}(\lambda, \nu(\lambda) ; 0)=0, \quad \forall \lambda . \tag{4.110}
\end{equation*}
$$

As a result:

$$
\begin{equation*}
|\nu(\lambda)|=1 . \tag{4.111}
\end{equation*}
$$

Hence, we are looking for solutions of:

$$
\begin{equation*}
|\nu(\lambda)|=1 \quad \Rightarrow \quad \lambda \in \Lambda(p, 0) . \tag{4.112}
\end{equation*}
$$

The roots can be numerically explored as follows: Noting that $\nu(0)=1$, we can construct a path of discrete points $\lambda_{k}, k=1,2, \ldots$, in the complex plane on which $\left|\nu\left(\lambda_{k}\right)\right|=1$ for all k. We require that $\left|\lambda_{k+1}-\lambda_{k}\right|=\Delta \lambda$, with $\Delta \lambda>0$, which implies that $\lambda_{k+1}$ will be located on a circle of radius $\Delta \lambda$ around $\lambda_{k}$. Assuming that $\Delta \lambda$ is chosen sufficiently small to ensure that only one solution for $\lambda_{k+1} \neq \lambda_{k-1}$ exists, a binary search on part of the circle is adequate to find the desired solution.

We do not have to explore the entire complex $\lambda$-plane because, if $\lambda$ is a solution, then also its complex conjugate, $\bar{\lambda}$, is a solution. This can be shown as follows: $f_{p}$ is a polynomial of $\lambda$ of degree $p+1$ and can therefore be written as:

$$
\begin{equation*}
f_{p}(\lambda, \mu ; 0)=\sum_{k=0}^{p+1} a_{k}(\mu) \lambda^{k} \tag{4.113}
\end{equation*}
$$

where $a_{k}$ can be written as:

$$
\begin{equation*}
a_{k}(\mu)=A_{k}+B_{k} \mu, \quad A_{k}, B_{k} \in \mathbb{R} \tag{4.114}
\end{equation*}
$$

Suppose now that $\lambda$ and $\mu$ satisfy:

$$
\begin{equation*}
f_{p}(\lambda, \mu ; 0)=0 \tag{4.115}
\end{equation*}
$$

Then

$$
\begin{align*}
f_{p}(\bar{\lambda}, \bar{\mu} ; 0) & =\sum_{k=0}^{p+1}\left(A_{k}+B_{k} \bar{\mu}\right) \bar{\lambda}^{k} \\
& =\sum_{k=0}^{p+1} \overline{\left(A_{k}+B_{k} \mu\right) \lambda^{k}} \\
& =\overline{f_{p}(\lambda, \mu ; 0)}=0 \tag{4.116}
\end{align*}
$$

Hence

$$
\begin{equation*}
f_{p}(\lambda, \mu ; 0)=0 \Leftrightarrow f_{p}(\bar{\lambda}, \bar{\mu} ; 0)=0 \tag{4.117}
\end{equation*}
$$

In Fig.(4.3) the above situation has been depicted.


Figure 4.3: Eigenvalues $\lambda_{k}$ and $\bar{\lambda}_{k}$ in the complex $\lambda$-plane. Eigenvalues are presented for $\theta \in\left[0, \frac{\pi}{2}\right]$, where $\mu=e^{i \theta}$ and $\theta=k \Delta x$.

## General case $\epsilon>0$

We are looking for the set $\Lambda(p, \epsilon)=0$. We write:

$$
\begin{equation*}
\phi_{k}^{p}(\lambda, \mu ; \epsilon)=\alpha_{k}^{p}(\lambda ; \epsilon)+\epsilon \mu \beta_{k}^{p}(\lambda), \tag{4.118}
\end{equation*}
$$

where

$$
\begin{align*}
\alpha_{k}^{p}(\lambda ; \epsilon)= & \alpha_{k+2}^{p}(\lambda ; \epsilon)-\frac{2 k+1}{\lambda}\left[2 \alpha_{k+1}^{p}(\lambda ; \epsilon)-(1+\epsilon)\right], \\
& \alpha_{p+2}^{p}(\lambda ; \epsilon)=\alpha_{p+1}^{p}(\lambda ; \epsilon)=0,  \tag{4.119}\\
\beta_{k}^{p}(\lambda)= & \beta_{k+2}^{p}(\lambda)-\frac{2 k+1}{\lambda}\left[2 \beta_{k+1}^{p}(\lambda)+(-1)^{k}\right], \\
& \beta_{p+2}^{p}(\lambda)=\beta_{p+1}^{p}(\lambda)=0, \tag{4.120}
\end{align*}
$$

such that $\phi_{k}^{p}$ given by Eq.(4.118) indeed satisfies Eq.(4.106). It is noted that:

$$
\begin{equation*}
\phi_{k}^{p}(\lambda, \mu ; \epsilon)=(1+\epsilon) \phi_{k}^{p}(\lambda, \mu ; 0)+\epsilon \mu \beta_{k}^{p}(\lambda) \tag{4.121}
\end{equation*}
$$

The characteristic polynomial of Eq.(4.105) can now be rewritten as:

$$
\begin{equation*}
c_{2}(\lambda ; \epsilon) \mu^{2}+c_{1}(\lambda ; \epsilon) \mu+c_{0}(\lambda ; \epsilon)=0 \tag{4.122}
\end{equation*}
$$

where

$$
\begin{align*}
& c_{2}(\lambda ; \epsilon) \equiv \epsilon\left(\beta_{1}^{p}-\beta_{0}^{p}\right),  \tag{4.123}\\
& c_{1}(\lambda ; \epsilon) \equiv 1+\alpha_{0}^{p}-\alpha_{1}^{p}+\epsilon\left(\beta_{0}^{p}-\beta_{1}^{p}\right),  \tag{4.124}\\
& c_{0}(\lambda ; \epsilon) \equiv \alpha_{0}^{p}-\alpha_{1}^{p} . \tag{4.125}
\end{align*}
$$

Now define:

$$
\begin{equation*}
\nu^{ \pm}(\lambda ; \epsilon)=\frac{-c_{1} \pm \sqrt{c_{1}^{2}-4 c_{2} c_{0}}}{2 c_{0}} \tag{4.126}
\end{equation*}
$$

which satisfies:

$$
\begin{equation*}
f_{p}\left(\lambda, \nu^{ \pm}(\lambda ; \epsilon) ; \epsilon\right)=0 \quad \forall \lambda \tag{4.127}
\end{equation*}
$$

As a result:

$$
\begin{equation*}
\left|\nu^{ \pm}(\lambda ; \epsilon)\right|=1 \quad \Rightarrow \quad \lambda \in \Lambda(p, \epsilon) . \tag{4.128}
\end{equation*}
$$

Hence we are looking for solutions of:

$$
\begin{equation*}
\left|\nu^{ \pm}(\lambda ; \epsilon)\right|=1 \tag{4.129}
\end{equation*}
$$

When we want to check wether $\left|\nu^{ \pm}\right|=1$ for a specific value of $\lambda$, this is complicated because $\nu^{+}$and $\nu^{-}$are discontinuous in $\lambda$ due to so-called branch points. To explore the roots of the characteristic polynomial it is therefore not sufficient to simply consider Eq.(4.129). Instead we consider for $\epsilon>0$ the function:

$$
\begin{equation*}
F(\lambda ; \epsilon) \equiv\left(\left|\nu^{+}(\lambda ; \epsilon)\right|-1\right)\left(\left|\nu^{-}(\lambda ; \epsilon)\right|-1\right) \tag{4.130}
\end{equation*}
$$

Now, when $F(\lambda ; \epsilon)=0$ then either

$$
\begin{equation*}
\left|\nu^{+}(\lambda ; \epsilon)\right|=1 \quad \text { or } \quad\left|\nu^{-}(\lambda ; \epsilon)\right|=1 \tag{4.131}
\end{equation*}
$$

Since also

$$
\begin{equation*}
f_{p}\left(\lambda, \nu^{+}(\lambda ; \epsilon) ; \epsilon\right)=0 \quad \text { and } \quad f_{p}\left(\lambda, \nu^{-}(\lambda ; \epsilon) ; \epsilon\right)=0 \tag{4.132}
\end{equation*}
$$

we have:

$$
\begin{equation*}
F(\lambda ; \epsilon)=0 \quad \Rightarrow \quad \lambda \in \Lambda(p, \epsilon) \tag{4.133}
\end{equation*}
$$

Hence, we are looking for solutions of $F(\lambda ; \epsilon)=0$.
The roots are numerically explored in a similar fashion as the roots for the special case $\epsilon=0$.

### 4.3.4 Results

In this section the following definitions will be used extensively:

$$
\begin{equation*}
\lambda^{*} \equiv \frac{\lambda}{\pi(p+1)}, \quad k^{*} \equiv \frac{k}{\pi(p+1)} . \tag{4.134}
\end{equation*}
$$

We will call $\lambda^{*}$ and $k^{*}$ the scaled eigenvalue (or root) and scaled wavenumber, respectively. The wavenumber $k$ is related to the wavelength $l$ by the relation $k=\frac{2 \pi}{l}$. The shortest wavelength which can be represented on the (uniform) mesh is given by:

$$
\begin{equation*}
l_{\min }=\frac{2 \Delta x}{p+1} \tag{4.135}
\end{equation*}
$$

hence, the maximum wavenumber is given by:

$$
\begin{equation*}
k_{\max }=\frac{2 \pi}{l_{\min }} \Rightarrow\left(k^{*} \Delta x\right)_{\max }=1 \tag{4.136}
\end{equation*}
$$

Throughout this section it is furthermore assumed that $a=1$ in Eq.(4.2).

## Results for $\epsilon=0$

Fig.(4.4) shows the (scaled) numerical dispersion relation for $\epsilon=0$ for $p=0$ up to $p=5$, presented in a way which is quite common in literature, see for example [43], [77]. The slopes of the contours represent the velocities with which the numerically represented waves propagate. In Fig.(4.5) the scaled numerical dissipation is presented as a function of the scaled wavenumber. In Figs.(4.4) and (4.5) only the results for $k^{*} \Delta x \in[0,1]$ have been presented. The results for $k^{*} \Delta x \in[-1,0]$ for the numerical dispersion and dissipation are obtained from the presented results by taking the complex conjugate of the results. It is observed from Fig.(4.4) that for higher orders $p$ the exact dispersion relation is better approximated for a larger range of (scaled) wavenumbers. Fig.(4.5) shows that dissipation is less for higher orders of $p$ for a larger range of (scaled) wavenumbers as well. After a certain wavenumber the dissipation increases rapidly. The maximum dissipation, obtained for $k^{*} \Delta x=1$, becomes larger (in magnitude) with increasing $p$.

In Figs.(4.4) and (4.5), for a given degree $p$, no distinction has been made between the contribution of the different eigenvalues $\lambda_{k}^{*}$ to the contours in the complex $\lambda^{*}$-plane. The presented contours are obtained from combining the results of all $\lambda_{k}^{*}, k \in\left[1, \frac{1}{2}(p+1)\right]$, as is also depicted in Fig.(4.3). Note that the eigenvalues $\lambda_{k}^{*}, k=4,5,6$, are the complex conjugate of the shown eigenvalues $\lambda_{1}^{*}, \lambda_{2}^{*}$ and $\lambda_{3}^{*}$, respectively. When $p$ is even, say for example 4 , the contour is obtained by presenting $\lambda_{1}^{*}, \lambda_{2}^{*}, \lambda_{3}^{*}$ and a part (half) of $\lambda_{4}^{*}$ (for $k^{*} \in$ $[0, \pi / 2]$ ). Figs.(4.6) and (4.7) show, respectively, the numerical dispersion and dissipation, again for $p=5$, however, this time the contribution of the different roots has been made explicit, which in literature is often omitted.

A drawback of presenting the numerical dispersion relation graphically as in Figs.(4.4) and (4.6), is that it is not clear for which values of $k \Delta x$ the different roots (principle and spurious) exist. From Fig.(4.6) the ranges for $k^{*} \Delta x$ can be obtained for which the roots labelled "principle", "spurious \#1" and "spurious \#2" are given. These ranges have been presented in table (4.1) as well. The values for $k \Delta x$, which are also presented in table (4.1), are derived from the ranges $k^{*} \Delta x$ employing Eq.(4.134). The ranges $k^{*} \Delta x$ show that the principle and two spurious roots exist for consecutive ranges of $k^{*} \Delta x$, which might suggest that the roots are not present simultaneously. However, the principle and spurious roots do exist simultaneously for $k \Delta x \in[0,2 \pi]$, which was already shown graphically by Fig.(4.3) and furthermore explained in the text of section 4.3.3. The values of $k \Delta x$ presented in table (4.1) are all (integer) multiples of the range $k \Delta x \in[0,2 \pi]$. With $\mu=e^{i k \Delta x}$, all these three ranges result in $\mu$ describing the same circle in the complex plane once. The results for the three roots must all be interpreted in the range $[0,2 \pi]$. From Eq.(4.8) it can now be concluded that the eigenvalue called "principle" closely approximates the exact dispersion relation throughout the range of wavenumbers $k \Delta x \in[0,2 \pi]$. The eigenvalue called 'spurious \#1" only approximates the exact dispersion relation quite well for small wavenumbers. For larger wavenumbers the velocity of the wave represented by the wave belonging to this eigenvalue, deviates more and more from the physical propagation velocity $a=1$. The wave numerically represented by the eigenvalue called "spurious \#2" propagates with an unphysical velocity for all wavenumbers. For wavenumbers above, approximately, $k^{*}=0.83$ the numerically represented wave even propagates in the opposite direction with a velocity that is much larger in magnitude than $a$.


Figure 4.4: Numerical dispersion relation for $\epsilon=0$ for $p=0$ up to $p=5$. The dotted line represents the exact dispersion relation $\operatorname{Im}\left(\lambda^{*}\right)=k^{*} \Delta x$.


Figure 4.5: Numerical dissipation for $\epsilon=0$ for $p=0$ up to $p=5$. The exact dissipation relation is given by $\operatorname{Re}\left(\lambda^{*}\right)=0$.

An alternative and less ambiguous way of presenting the results is shown in Fig.(4.8). It shows the scaled eigenvalues $\lambda_{k}^{*}$ in the complex $\lambda^{*}$-plane for $p=0$ up to $p=5$. In this graph the exact dispersion relation coincides with the vertical axis. Note that the figure is presented with an adjusted horizontal plotting range. It is observed from Fig.(4.8) that with


FIGURE 4.6: Numerical dispersion relation for $\epsilon=0$ for $p=5$, where the contour is given by the three eigenvalues $\lambda_{k}^{*}, k=1,2,3$. Note that the eigenvalues $\lambda_{k}^{*}, k=4,5,6$, are the complex conjugate of the shown eigenvalues $\lambda_{1}^{*}$, $\lambda_{2}^{*}$ and $\lambda_{3}^{*}$, respectively.


FIGURE 4.7: Numerical dissipation for $\epsilon=0$ for $p=5$, where the contour is given by the three eigenvalues $\lambda_{k}^{*}$, $k=1,2,3$. Note that the eigenvalues $\lambda_{k}^{*}, k=4,5,6$, are the complex conjugate of the shown eigenvalues $\lambda_{1}^{*}, \lambda_{2}^{*}$ and $\lambda_{3}^{*}$, respectively.

| $\mathrm{p}=5$ | root | range $k^{*} \Delta x$ | range $k \Delta x$ |
| :--- | :--- | :---: | :---: |
|  | principle | $0-\frac{1}{3}$ | $0-2 \pi$ |
|  | spurious \#1 | $\frac{1}{3}-\frac{2}{3}$ | $2 \pi-4 \pi$ |
|  | spurious \#2 | $\frac{2}{3}-1$ | $4 \pi-6 \pi$ |

TAble 4.1: Ranges $k^{*} \Delta x$ for which the principle, spurious \#1 and spurious \#2 are given in Fig.(4.6) for $p=5$. The values $k \Delta x$ are derived from these values.
increasing $p$, the principle root approximates the exact dispersion relation increasingly well; it also becomes relatively smaller in length. Fig.(4.9) shows the scaled eigenvalues $\lambda_{k}^{*}$ in the complex $\lambda^{*}$-plane for $p=12$. The figure is presented with an adjusted horizontal plotting range to better visualize $\lambda_{1}$ to $\lambda_{4}$. From Fig.(4.9) it is concluded that at least $\lambda_{1}, \lambda_{2}$ and $\lambda_{3}$ approximate the exact dispersion relation well, and can therefore be labelled as principle roots or modes. Even $\lambda_{4}$ only deviates little from the exact dispersion relation. For $p=12$ $\lambda_{5}, \lambda_{6}$ and $\lambda_{7}$ are spurious. Especially $\lambda_{6}$ and $\lambda_{7}$ have large dissipation.

Fig.(4.10) shows the scaled eigenvalues for larger values of $p$, up to $p=100$. For (very) large values of $p$ the length of the curve represented by a root $\lambda_{k}$ becomes very small, possibly smaller even than $\Delta \lambda$ ( $\Delta \lambda$ has been introduced in section 4.3.3 describing the numerical solution of the characteristic polynomial. $\Delta \lambda$ is the distance between two discrete points in the complex $\lambda$ plane.). Therefore, for these large values of $p$ it becomes very difficult to distinguish between the different roots $\lambda_{k}$. In Fig.(4.10) only the complete contour is shown, the contribution to the contour of the different parts $\lambda_{k}$ has not been presented. For larger values of $p$ it is more convenient to investigate dispersion and dissipation in terms of an error. Employing Eq.(4.88), for each wave corresponding to eigenvalue $\lambda_{k}$ we can compare:

$$
\begin{equation*}
e^{i j k \Delta x} e^{-\lambda_{m} \frac{a t}{\Delta x}} \tag{4.137}
\end{equation*}
$$

obtained for the numerical waves, with the exact waveform of Eq.(4.82):

$$
\begin{equation*}
e^{i j k \Delta x} e^{-i \omega t} \tag{4.138}
\end{equation*}
$$

Upon dividing Eq.(4.137) by Eq.(4.138) and upon using the exact dispersion relation $\omega=a k$, we obtain:

$$
\begin{equation*}
e^{\left(-\left[\operatorname{Re}\left(\lambda_{m}\right)+i \operatorname{Im}\left(\lambda_{m}\right)\right]+i k \Delta x\right) \frac{a t}{\Delta x}} \tag{4.139}
\end{equation*}
$$

Let us, suggested by Eq.(4.139), define the dispersion error $\delta_{i}$ and dissipation error $\delta_{r}$ as:

$$
\begin{equation*}
\delta_{i} \equiv 1-\left|e^{\operatorname{Im}\left(\lambda_{m}\right)-k \Delta x}\right|, \quad \delta_{r} \equiv 1-\left|e^{\operatorname{Re}\left(\lambda_{m}\right)}\right| \tag{4.140}
\end{equation*}
$$

From Eq.(4.140) scaled errors can be derived employing Eq.(4.134). Fig.(4.11) shows the scaled dispersion error $\delta_{i}^{*}$ as a function of the scaled wavenumber $k^{*} \Delta x$ for $p=0,1,2,5$, $10,20,50$ and $p=100$. It is concluded from Fig.(4.11) that the slopes become increasingly steep with increasing order $p$, as might have been expected. For $p>1$ the error is not visible for low values of $k^{*} \Delta x$ in the graph. This because the evaluated error is of machine accuracy. Fig.(4.12) shows the dissipation error as function of the scaled wavenumber for the same values of $p$. In Figs.(4.13) and (4.14) the dispersion and dissipation errors have


Figure 4.8: Scaled eigenvalues $\lambda_{k}^{*}$ in complex $\lambda^{*}$-plane for $p=0$ up to $p=5$, where the contribution of the different eigenvalues $\lambda_{k}^{*}$ has been visualized.


Figure 4.9: Scaled eigenvalues $\lambda_{k}^{*}$ in complex $\lambda^{*}$-plane for $p=12$.
been presented for $p=12$, respectively, where this time the contribution to the contour of the different eigenvalues has been highlighted. In Fig.(4.13) we observe for $\lambda_{1}$ a few
$\qquad$


Figure 4.10: Scaled eigenvalues $\lambda_{k}^{*}$ in complex $\lambda^{*}$ plane for $p=10, p=20, p=50$ and $p=100$.
spikes of machine accuracy. For every $p$ the first eigenvalue, $\lambda_{1}$, is most accurate, the second eigenvalue is more accurate than the third one and so on. In Fig.(4.15) the obtained values for the slopes for the dispersion and dissipation error are presented. The values of the slopes presented in the table have been obtained for small wavenumbers. Only for the first few values of $p$ we can be certain that the obtained slopes are the slopes for the first eigenvalue $\lambda_{1}$. For larger values of $p$ the errors, $\delta_{i}^{*}$ and $\delta_{r}^{*}$, for $\lambda_{1}$ were found to be of machine accuracy. The presented values of the slopes for larger values of $p$ are therefore not obtained for the most accurate eigenvalue $\lambda_{1}$. The slopes for the dispersion error, $\delta_{i}^{*}$, for at least $p=0$ up to $p=3$ suggest that the dispersion error decreases at order $2 p+3$. For the dissipation error, $\delta_{r}^{*}$, the dissipation error is found to decrease at order $2 p+2$. It means that the first eigenvalue, $\lambda_{1}$, is accurate to order $2 p+2$ (measure is given by the real part of the eigenvalues, which is related to $\delta_{r}^{*}$ ). This order was also found by Lowrie, [57]. However, it was found by Rasetarinera, Hussaini and Hu [65], that both the dissipation and dispersion errors of the discontinuous Galerkin method decay at order $2 p+2$ when the exact characteristic splitting formula is used (which corresponds to $\epsilon=0$ in our formulation).

In Eq.(4.89) the notation $\mu \equiv e^{i k \Delta x}$ was introduced and in Eq.(4.87) $\lambda=\frac{i \omega \Delta x}{a}$. From the exact dispersion relation $\omega=a k$, Eq.(4.79), we have:

$$
\begin{equation*}
e^{\lambda}=\mu \tag{4.141}
\end{equation*}
$$

For $\epsilon=0$ we replaced $\mu$ in Eq.(4.109) by:

$$
\nu(\lambda) \equiv \frac{\phi_{1}^{p}(\lambda ; 0)-\phi_{0}^{p}(\lambda ; 0)}{1+\phi_{0}^{p}(\lambda ; 0)-\phi_{1}^{p}(\lambda ; 0)} .
$$



Figure 4.11: Scaled dispersion error for $p=0,1,2,5$, $10,20,50$ and $p=100$.


Figure 4.12: Scaled dissipation error for $p=0,1,2,5$, $10,20,50$ and $p=100$.

Hence from the presented results we obtain, for $|\nu(\lambda)|=1$, the following hypothesis:

$$
\begin{equation*}
e^{\lambda}=\frac{\phi_{1}^{p}(\lambda ; 0)-\phi_{0}^{p}(\lambda ; 0)}{1+\phi_{0}^{p}(\lambda ; 0)-\phi_{1}^{p}(\lambda ; 0)}+\mathcal{O}\left(\lambda^{2 p+2}\right) . \tag{4.142}
\end{equation*}
$$

It was found, employing Maple, that, for the first few values of $p$, Eq.(4.142) results exactly


Figure 4.13: Scaled dispersion error for $p=12$, where $\lambda_{m}^{*}$ is represented by a solid line if $m$ is odd and by a dotted line if $m$ is even.


Figure 4.14: Scaled dissipation error for $p=12$, where $\lambda_{m}^{*}$ is represented by a solid line if $m$ is odd and by a dotted line if $m$ is even.
in a Padé approximation of $e^{i k \Delta x}$, which is of order $2 p+2$. In [42], Hu and Atkins mention that they verified the Padé expansion for $p$ up to 16 .

| $\epsilon=0$ |  |  |
| ---: | ---: | ---: |
| p | slope $\delta_{i}^{*}$ | slope $\delta_{r}^{*}$ |
| 0 | 3.000 | 2.000 |
| 1 | 5.000 | 4.000 |
| 2 | 7.001 | 5.999 |
| 3 | 8.975 | 7.994 |
| 4 | 10.960 | 9.980 |
| 5 | 13.003 | 11.949 |
| 6 | 14.813 | 13.898 |
| 7 | 17.014 | 15.817 |
| 8 | 18.158 | 17.720 |
| 9 | 20.407 | 19.556 |
| 10 | 21.869 | 21.406 |



FIGURE 4.15: The presented table shows obtained values for the slopes of the dispersion $\operatorname{error}\left(\delta_{i}^{*}\right)$ and dissipation error $\left(\delta_{r}^{*}\right)$ for $p=0$ up to $p=10$ for $\epsilon=0$. The figure gives a graphical representation of these values, where the slope of the dispersion error and dissipation error are denoted by $\delta_{i}^{*^{\prime}}$ and $\delta_{r}^{*^{\prime}}$, respectively.

## $\underline{\text { Results for } \epsilon>0}$

Fig.(4.16) shows the scaled eigenvalues $\lambda_{k}^{*}$ in complex $\lambda^{*}$-plane for $p=0$ up to $p=5$, for $\epsilon=0$ and $\epsilon=0.05$. The case $\epsilon=0$ corresponds to the algorithm employing exact flux solutions at element interfaces. For $\epsilon>0$ the flux at element interfaces is approximated by the Lax-Friedrichs flux. In Fig.(4.16) we used $\epsilon=0.05$, a value which is representative of the values used in the 3D unstructured-grid method. Clearly the scaled eigenvalues $\lambda_{k}^{*}$ for $\epsilon=0.05$ have larger damping, as is expected. The damping was found to increase with increasing $\epsilon$.

For $\epsilon>0$ the dispersion and dissipation errors, see Eqs.(4.140), can be evaluated. In Fig.(4.17) the obtained values of the slopes for the dispersion and dissipation error are presented. In this table only the results for $p=0$ up to $p=4$ are presented. The procedure for obtaining the eigenvalues for $\epsilon>0$ is more complicated, and for higher values of $p$ the obtained values for the slopes become less reliable. The presented slope values have been obtained for $\lambda_{1}$.

### 4.4 Analysis of fully-dicrete scheme

In the present section the dissipation and dispersion error analysis of the full-discretization of the one-dimensional scalar advection equation, see section 6.29 , is presented. In addition, stability of the schemes is investigated. The analysis is performed assuming that the domain $\Omega$ is now given by $\Omega=\mathbb{R}$ and that $I_{t} \in \mathbb{R}$, where both $\Omega$ and $I_{t}$ have been introduced in section 4.2.1.


Figure 4.16: Scaled eigenvalues $\lambda_{k}^{*}$ in complex $\lambda^{*}$ plane for $p=0$ up to $p=5$, for both $\epsilon=0$ and $\epsilon=0.05$.

| $\epsilon=0.05$ |  |  |
| ---: | ---: | ---: |
| p | slope $\delta_{i}^{*}$ | slope $\delta_{r}^{*}$ |
| 0 | 3.000 | 2.000 |
| 1 | 5.000 | 4.000 |
| 2 | 7.000 | 5.998 |
| 3 | 9.007 | 8.002 |
| 4 | 10.983 | 9.963 |



Figure 4.17: The presented table shows obtained values for the slopes of the dispersion error $\left(\delta_{i}^{*}\right)$ and dissipation error $\left(\delta_{r}^{*}\right)$ for $p=0$ up to $p=4$ for $\epsilon=0.05$. The figure gives a graphical representation of these values, where the slope of the dispersion error and dissipation error are denoted by $\delta_{i}^{*^{\prime}}$ and $\delta_{r}^{*^{\prime}}$, respectively.

### 4.4.1 Dispersion relation of fully-discrete system

In order to analyze the stability of the fully-discrete system the approach presented in section 4.3.1 is extended to the fully-discrete system or scheme. We substitute:

$$
\begin{equation*}
\overline{\mathbf{v}}_{j}^{n}=\hat{\mathbf{v}} e^{i(j k \Delta x-n \omega \Delta t)} \tag{4.143}
\end{equation*}
$$

in Eq.(4.68), where $L_{f}$ is either given by Eq.(4.71), Eq.(4.73) or Eq.(4.77), corresponding to Euler explicit, Euler implicit or Runge-Kutta, respectively. Substitution of Eq.(4.143) into Eq.(4.68) leads to:

$$
\begin{equation*}
\left[P\left(e^{-i k \Delta x}, e^{i k \Delta x}\right) e^{-i \omega \Delta t}-Q\left(e^{-i k \Delta x}, e^{i k \Delta x}\right)\right] \hat{\mathbf{v}} e^{i(j k \Delta x-n \omega \Delta t)}=\mathbf{0} . \tag{4.144}
\end{equation*}
$$

Since Eq.(4.144) holds for all $j$, a non-trivial solution of Eq.(4.144) can be obtained if, and only if:

$$
\begin{equation*}
\operatorname{det}(Q-\tilde{\lambda} P)=0, \quad \tilde{\lambda} \equiv e^{-i \omega \Delta t} \tag{4.145}
\end{equation*}
$$

Eq.(4.145) is the dispersion relation of the fully-discrete system, and $\tilde{\lambda}$ is an eigenvalue of the matrix $P^{-1} Q . P$ and $Q$ can be functions of $\bar{K}$ and, in addition, from the CFL number, as can be observed from Eq.(4.71), Eq.(4.73) and Eq.(4.77). That means that we can write for Eq.(4.145):

$$
\begin{equation*}
\operatorname{det}(\mathcal{P}(\sigma \bar{K})-\tilde{\lambda} I)=0 \tag{4.146}
\end{equation*}
$$

where the matrix $\mathcal{P}(\sigma \bar{K})$ is a polynomial in $\sigma \bar{K}$, the CFL number is given by $\sigma=\frac{a \Delta t}{\Delta x}$. The spectral mapping theorem of Varga [40] then states that:

$$
\begin{equation*}
\tilde{\lambda}=\mathcal{P}(\sigma \lambda) \tag{4.147}
\end{equation*}
$$

where $\lambda$ are the eigenvalues of $\bar{K}$. This can be shown as follows; Write

$$
\begin{equation*}
\mathcal{P}(\sigma \bar{K})=\sum_{k=0}^{N} a_{k}(\sigma \bar{K})^{k} \tag{4.148}
\end{equation*}
$$

and let $\mathbf{v}$ be an eigenvector of $\bar{K}$, corresponding to the eigenvalue $\lambda$ :

$$
\begin{equation*}
\bar{K} \mathbf{v}=\lambda \mathbf{v} \tag{4.149}
\end{equation*}
$$

Since, for example,

$$
\begin{equation*}
(\sigma \bar{K})^{2} \mathbf{v}=\sigma^{2} \bar{K}(\bar{K} \mathbf{v})=\sigma^{2} \bar{K} \lambda \mathbf{v}=\sigma^{2} \lambda \bar{K} \mathbf{v}=\sigma^{2} \lambda^{2} \mathbf{v} \tag{4.150}
\end{equation*}
$$

the vector $\mathbf{v}$ is also an eigenvector of $\mathcal{P}(\bar{K})$ :

$$
\begin{equation*}
\mathcal{P}(\sigma \bar{K}) \mathbf{v}=\left(\sum_{k=0}^{N} a_{k}(\sigma \bar{K})^{k}\right) \mathbf{v}=\left(\sum_{k=0}^{N} a_{k}(\sigma \lambda)^{k}\right) \mathbf{v} \tag{4.151}
\end{equation*}
$$

with corresponding eigenvalue:

$$
\begin{equation*}
\tilde{\lambda}=\sum_{k=0}^{N} a_{k}(\sigma \lambda)^{k}=\mathcal{P}(\sigma \lambda) \tag{4.152}
\end{equation*}
$$

The spectral mapping theorem greatly reduces the amount of work necessary to obtain the eigenvalues. In section 4.3 the eigenvalues of $\bar{K}$ were already obtained.

In order to prevent numerical modes to grow faster than physical modes, we impose the following stability condition ([40]):

$$
\begin{equation*}
|\tilde{\lambda}| \leq 1 \tag{4.153}
\end{equation*}
$$

From the stability condition we obtain the maximum CFL-number:

$$
\begin{equation*}
\sigma_{\max }=\left(\frac{a \Delta t}{\Delta x}\right)_{\max } \tag{4.154}
\end{equation*}
$$

such that:

$$
\begin{equation*}
\tilde{\lambda}=\mathcal{P}\left(\sigma_{\max } \lambda\right) \quad \Rightarrow \quad|\tilde{\lambda}|=1 \tag{4.155}
\end{equation*}
$$

### 4.4.2 Stability

Often the reason for conducting an analysis of the fully-discrete scheme is to obtain the maximum allowable time-step $\Delta t$, or actually maximum CFL-number $\sigma_{\max }$. Most schemes become unstable when the time step is increased above a certain time-step $(\Delta t)_{\max }$, which means that the numerical result will grow unphysically and unbounded, rendering the result useless.

In this section we will derive the stability criterion, $\sigma_{\max }$, as follows; First we derive from Eq.(4.146), employing the spectral mapping theorem, the contour $\lambda_{\max }$ for which $|\tilde{\lambda}|=1$ :

$$
\begin{equation*}
|\tilde{\lambda}|=1, \quad \tilde{\lambda}=\mathcal{P}\left(\lambda_{\max }\right)=a_{0}+a_{1} \lambda_{\max }+\ldots+a_{N} \lambda_{\max }^{N} \tag{4.156}
\end{equation*}
$$

Then, by considering the eigenvalues $\lambda$ of the semi-discrete scheme, we can derive the factor $\sigma$ with which we have to multiply $\lambda$ in order to make $\sigma \lambda$ fit within the contour given by $|\tilde{\lambda}|=1$.

The contour $|\tilde{\lambda}|=1$ can be obtained in a similar way as the contours for the eigenvalues $\lambda$ in section 4.3.3.

## Results

Fig.(4.18) shows the stability contour $(|\tilde{\lambda}|=1)$ for the Euler explicit time-integration. The region in which the fully-discrete scheme can be stable is limited. Fig.(4.19) shows the stability contour for the Euler implicit time-integration. The stability region is now located outside of the contour given by $|\tilde{\lambda}|=1$ (the contour can be shown to be a circle), meaning that the region, in which the fully discrete scheme employing the Euler implicit time-integration is stable, is huge.

Fig.(4.20) and Fig.(4.21) show the stability contours $|\tilde{\lambda}|=1$ in the complex $\tilde{\lambda}$-plane for $N \in[1,10]$ and $N \in[11,20]$, respectively, where the coefficients in the Runge-Kutta scheme of Eq.(4.77) are taken $\gamma_{N}=1, \gamma_{N-1}=\frac{1}{2}, \ldots \gamma_{1}=\frac{1}{N}$. The region in which the fullydiscrete scheme is stable lies within the contours and it grows with increasing number of stages $N$.

In Fig.(4.20) we observe little "islands of stability" for $\operatorname{Re}(\tilde{\lambda})>0$. With increasing $N$ the 'islands"become smaller and move away from the origin of the coordinate system. Fig.(4.21) even shows two groups of '"slands of stability".


Figure 4.18: Stability region in the complex $\lambda$-plane for the Euler explicit time-integration.


Figure 4.19: Stability region in the complex $\lambda$-plane for the Euler implicit time-integration.

For the fully-discrete system employing the Euler explicit time-integration to be stable, the contour $\sigma \lambda$, where $\lambda$ is the eigenvalue of $\bar{K}$, has to fit entirely in the region denoted by "stable". If this requirement is not satisfied the scheme is unstable for that timestep.


Figure 4.20: Stability region in the complex $\lambda$-plane for the $N$-stage Runge-Kutta time-integration, for $N \in$ [ 1,10 ].


Figure 4.21: Stability region in the complex $\lambda$-plane for the $N$-stage Runge-Kutta time-integration, for $N \in$ [11, 20].

# Convection of a 2D GaUSSIAN PULSE 

### 5.1 Introduction

After developing a new algorithm it is of paramount importance to verify the algorithm. To this end, so-called, verification problems have been considered. With a verification problem we mean a problem of which the analytical solution is known or can be derived and to which numerically obtained results can be compared. One of these problems is the subject of this chapter; the convection of a two-dimensional Gaussian pulse.

The choice for a two-dimensional problem was motivated by the following: within the development of the computational method, the problem was the first (major) verification problem to be considered. At that time (spring 2000) the algorithm was second-order accurate (in space) and not yet parallelized. Furthermore, the characteristic-based non-reflecting boundary conditions, which were used at the in and outflow boundaries, are approximations and are known to produce some reflections. To avoid substantial contamination of the numerical results by these reflections, the boundaries of the computational domain are in the far field, leading to a three-dimensional computational domain of considerable size. The large domain could be divided into a relatively coarse mesh only to enable computations on a single processor. However, for a good comparison between numerical and analytical results the finest affordable mesh would still be too coarse. A two-dimensional problem leads to a reduction of the computational demands.

The results, which will be presented in this chapter, have been obtained with the parallelized version ${ }^{1}$ of the computational method, which has been implemented on TERAS. We were granted access to TERAS by the NCF (National Computing Facilities). TERAS is a 1024 processor CPU platform, consisting of two 512-SPU SGI-Origin 3800 systems.

The basis functions used for the discontinuous Galerkin space discretization are polynomials of degree $\leq 1$ and the time integration is performed by the four stage, fourth order accurate, Runge-Kutta time integration algorithm (see chapter 3 for a description of the discontinuous Galerkin space discretization and Runge-Kutta time integration).

In section 5.2 a short description of the problem at hand is given. In the subsequent section, section 5.3, the analytical solution to the problem is derived. In section 5.4 the obtained numerical results are presented. In this section not only the comparison between the numerical results and the analytical solution are presented, also results of a performance test of the

[^3]parallelized algorithm on TERAS is presented.

### 5.2 Problem description

The dimensionless linearized Euler equations are solved on a square domain. Hereto, the linearized Euler equations, which were presented in chapter 2, are reduced to two spatial dimensions. An acoustic pulse is generated by an initial Gaussian distribution at the center of the computational domain $(x=y=0)$. The two-dimensional domain has dimensions $x \in[-100,100], y \in[-100,100]$, the mean or background flow is uniform with Machnumber components $M_{1}=M=0.5$ in $x$-direction and $M_{2}=0$ in $y$-direction, there are no sources ( $\mathbf{S}_{m}=\mathbf{S}_{i}=\mathbf{S}_{e}=\mathbf{0}$ ) and the initial condition for the 2D solution vector $\mathbf{u}(x, y, t)=\left(\rho^{\prime}, u^{\prime}, v^{\prime}, p^{\prime}\right)^{T}$ is given by:

$$
\mathbf{u}(x, y, 0)=\left(\begin{array}{c}
1  \tag{5.1}\\
\beta x \\
\beta y \\
1
\end{array}\right) e^{-\alpha\left(x^{2}+y^{2}\right)}
$$

with

$$
\begin{equation*}
\alpha=\frac{\ln (2)}{9}, \quad \beta=0.04 \tag{5.2}
\end{equation*}
$$

This test case has also been addressed by Atkins and Shu [6]. Together with a vorticity wave and an entropy pulse it is furthermore described as part of the ICASE/LaRC Workshop on Benchmark Problems in Computational Aeroacoustics [3]. Note, however, that in the benchmark-case the initial velocities, $u^{\prime}$ and $v^{\prime}$, are taken equal to zero. The original benchmark problem was developed to test non-reflecting boundary conditions. The characteristic based non-reflecting boundary conditions are known to result in (at least) some reflections which might contaminate the solution. Numerical computations are terminated before the introduced perturbation reaches the end of the computational domain, thus avoiding the effects of false reflections.

### 5.3 Analytical solution

Under the above conditions the dimensionless LEE simplify into (see also section 2.5):

$$
\begin{align*}
& \frac{\partial \rho^{\prime}}{\partial t}+M \frac{\partial \rho^{\prime}}{\partial x}+\frac{\partial u_{j}^{\prime}}{\partial x_{j}}=0  \tag{5.3}\\
& \frac{\partial u_{i}^{\prime}}{\partial t}+M \frac{\partial u_{i}^{\prime}}{\partial x}+\frac{\partial p^{\prime}}{\partial x_{i}}=0  \tag{5.4}\\
& \frac{\partial p^{\prime}}{\partial t}+M \frac{\partial p^{\prime}}{\partial x}+\frac{\partial u_{j}^{\prime}}{\partial x_{j}}=0 \tag{5.5}
\end{align*}
$$

clearly from Eqs.(5.3) and (5.5) we have $\rho^{\prime}=p^{\prime}$. In the remainder of this section Eq.(5.3) is dropped. By introducing the material derivative $\frac{D}{D t}=\frac{\partial}{\partial t}+M \frac{\partial}{\partial x}$ the LEE can be written in the form:

$$
\begin{equation*}
\frac{D u_{i}^{\prime}}{D t}+\frac{\partial p^{\prime}}{\partial x_{i}}=0 \tag{5.6}
\end{equation*}
$$

$$
\begin{equation*}
\frac{D p^{\prime}}{D t}+\frac{\partial u_{j}^{\prime}}{\partial x_{j}}=0 \tag{5.7}
\end{equation*}
$$

Taking the divergence of Eq.(5.6), and the $\frac{D}{D t}$ of Eq.(5.7), and subsequently subtracting the resulting equations results in the following convected wave equation for the pressure perturbation:

$$
\begin{equation*}
\frac{D^{2} p^{\prime}}{D t^{2}}-\frac{\partial^{2} p^{\prime}}{\partial x_{j} \partial x_{j}}=0 \tag{5.8}
\end{equation*}
$$

More general we have:

$$
\begin{equation*}
\frac{D^{2} q}{D t^{2}}-\frac{\partial^{2} q}{\partial x_{j} \partial x_{j}}=0, \quad \text { or } \quad \frac{D^{2} q}{D t^{2}}-\nabla^{2} q=0 \tag{5.9}
\end{equation*}
$$

where $q$ is either one of the primitive variables $\rho^{\prime}, u^{\prime}, v^{\prime}$ or $p^{\prime}$.
Upon introducing a coordinate system moving with the mean flow ([10]): $\tau=t, \xi=$ $x-M t, \eta=y$, we have:

$$
\begin{align*}
& \frac{\partial}{\partial t}=\frac{\partial}{\partial \tau}-M \frac{\partial}{\partial \xi}, \quad \frac{\partial}{\partial x}=\frac{\partial}{\partial \xi}, \quad \frac{\partial}{\partial y}=\frac{\partial}{\partial \eta}  \tag{5.10}\\
& \text { so } \frac{D}{D t}=\frac{\partial}{\partial \tau} \tag{5.11}
\end{align*}
$$

and the wave equation can be written as:

$$
\begin{equation*}
\frac{\partial^{2} q}{\partial \tau^{2}}-\frac{\partial q}{\partial \xi_{j} \partial \xi_{j}}=0, \quad \xi_{1}=\xi, \quad \xi_{2}=\eta \tag{5.12}
\end{equation*}
$$

In the remainder of this section we will solve Eq.(5.12) for the velocity potential $\phi$. The intial perturbation, Eq.(5.1), is of circular form, hence we introduce polar coordinates $(r, \theta)$, where $\xi=r \cos (\theta)$ and $\eta=r \sin (\theta)$ and solve Eq.(5.12) in polar coordinates. With $u^{\prime}=\frac{\partial \phi}{\partial x}$ and $v^{\prime}=\frac{\partial \phi}{\partial y}$ the primitive variables are related to the velocity potential in polar coordinates by

$$
\begin{equation*}
\frac{\partial \phi}{\partial r}=u_{r}^{\prime}, \quad \frac{\partial \phi}{\partial \tau}=-p^{\prime} \tag{5.13}
\end{equation*}
$$

where $u_{r}^{\prime}$ is the radial velocity component. (Note that for the azimuthal velocity we have $u_{\theta=0}^{\prime}$.) The initial conditions can be obtained from the relations

$$
\begin{align*}
& \phi(r, 0)=\int_{\infty}^{r} u_{r}^{\prime}(r) d r=-\frac{\beta}{2 \alpha} e^{-\alpha r^{2}}  \tag{5.14}\\
& \frac{\partial \phi}{\partial \tau}(r, 0)=-p^{\prime}(r, 0)=-e^{-\alpha r^{2}} \tag{5.15}
\end{align*}
$$

The wave equation for the velocity potential, which is independent of $\theta$, is now given by

$$
\begin{equation*}
\frac{\partial^{2} \phi}{\partial \tau^{2}}-\left(\frac{\partial^{2} \phi}{\partial r^{2}}+\frac{1}{r} \frac{\partial q}{\partial r}\right)=0 \tag{5.16}
\end{equation*}
$$

The solution of Eq.(5.16) with the initial conditions given by Eq.(5.14) and Eq.(5.15) can be obtained conveniently employing the Hankel transform (see for example [64] or [86]):

$$
\begin{align*}
& \phi(r, \tau)=\int_{0}^{\infty} \lambda J_{0}(\lambda r) \Phi(\lambda, \tau) d \lambda  \tag{5.17}\\
& \Phi(\lambda, \tau)=\int_{0}^{\infty} r J_{0}(\lambda r) \phi(r, \tau) d r \tag{5.18}
\end{align*}
$$

where $J_{0}$ is the zeroth-order Bessel function of the first kind. Upon applying the Hankel transform the problem reduces to solving

$$
\begin{equation*}
\frac{\partial^{2} \Phi}{\partial \tau^{2}}+\lambda^{2} \Phi=0 \tag{5.19}
\end{equation*}
$$

with

$$
\begin{equation*}
\Phi(\lambda, 0)=E(\lambda), \quad \frac{\partial \Phi}{\partial \tau}(\lambda, 0)=F(\lambda) \tag{5.20}
\end{equation*}
$$

where $E(\lambda)$ and $F(\lambda)$ are given by:

$$
\begin{align*}
& E(\lambda)=-\frac{\beta}{2 \alpha} \int_{0}^{\infty} r J_{0}(\lambda r) e^{-\alpha r^{2}} d r \\
& F(\lambda)=-\int_{0}^{\infty} r J_{0}(\lambda r) e^{-\alpha r^{2}} d r=\frac{2 \alpha}{\beta} E(\lambda) . \tag{5.21}
\end{align*}
$$

From Gradshteyn and Ryzhik [36] we obtain

$$
\begin{equation*}
\int_{0}^{\infty} r J_{0}(\lambda r) e^{-\alpha r^{2}} d r=\frac{1}{2 \alpha} e^{-\frac{\lambda^{2}}{4 \alpha}} \tag{5.22}
\end{equation*}
$$

The solution of Eq.(5.19) can be written as:

$$
\begin{equation*}
\Phi(\lambda, \tau)=E(\lambda) \cos (\lambda \tau)+\frac{F(\lambda)}{\lambda} \sin (\lambda \tau) \tag{5.23}
\end{equation*}
$$

Using Eq.(5.17) to carry out the inverse transform from $\lambda$ to $r$ we obtain for the velocity potential:

$$
\begin{equation*}
\phi(r, \tau)=-\frac{1}{2 \alpha} \int_{0}^{\infty}\left[\sin (\lambda \tau)+\frac{\beta}{2 \alpha} \lambda \cos (\lambda \tau)\right] J_{0}(\lambda r) e^{-\frac{\lambda^{2}}{4 \alpha}} d \lambda \tag{5.24}
\end{equation*}
$$

The general solution, as function of $r$ and $t$, finally becomes:

$$
\begin{align*}
p^{\prime}(r, t) & =\frac{1}{2 \alpha} \int_{0}^{\infty}\left[\cos (\lambda t)-\frac{\beta}{2 \alpha} \lambda \sin (\lambda t)\right] \lambda J_{0}(\lambda r) e^{-\frac{\lambda^{2}}{4 \alpha}} d \lambda,  \tag{5.25}\\
u^{\prime}(r, t) & =\frac{x-M t}{2 \alpha r} \int_{0}^{\infty}\left[\sin (\lambda t)+\frac{\beta}{2 \alpha} \lambda \cos (\lambda t)\right] \lambda J_{1}(\lambda r) e^{-\frac{\lambda^{2}}{4 \alpha}} d \lambda,  \tag{5.26}\\
v^{\prime}(r, t) & =\frac{y}{2 \alpha r} \int_{0}^{\infty}\left[\sin (\lambda t)+\frac{\beta}{2 \alpha} \lambda \cos (\lambda t)\right] \lambda J_{1}(\lambda r) e^{-\frac{\lambda^{2}}{4 \alpha}} d \lambda, \tag{5.27}
\end{align*}
$$

where

$$
\begin{equation*}
r=\sqrt{(x-M t)^{2}+y^{2}}, \tag{5.28}
\end{equation*}
$$

and $\rho^{\prime}=p^{\prime}$. In the above expressions $J_{1}$ is the first-order Bessel function of the first kind. At $r=0$ we obtain the exact solution for the pressure:

$$
\begin{equation*}
p^{\prime}(0, t)=1-\beta t+\left[1-\beta t+\frac{\beta}{2 \alpha}\right] i \sqrt{\alpha \pi} \operatorname{erf}(i t \sqrt{\alpha}) e^{-\alpha t^{2}} \tag{5.29}
\end{equation*}
$$

where erf is the error function. It should be noted that $i \operatorname{erf}(i t \sqrt{\alpha})$ is real.

### 5.4 Numerical result

For a correct implementation of the initial condition we have to project $\mathbf{u}(x, y, 0)$, given by Eq.(5.1), onto the basis functions. The integrations which then have to be performed are not straightforward. Alternatively, we will approximate the initial solution by a second-order accurate Taylor-series expansion around the centroid of each element as was also done by Atkins and Shu [6], see section 3.4 for a full description.

In order to perform this 2D calculation with the present 3D method all derivatives in the $z$ direction are taken equal to zero. Furthermore symmetry-plane boundary conditions are used for the upper and lower boundary in the $z$-direction. For the linear equations which we are considering, the symmetry-plane boundary condition is identical to the solid wall boundary condition, described in section 3.3. At all the other boundaries of the computational domain characteristic-based non-reflecting boundary conditions are used, see also section 3.3. The result in the plane $z=0$ is interpreted as the 2 D solution.

The simulations have been carried out on different tetrahedral meshes. The physical domain, $\Omega$, is partitioned into $N_{e}$ identical tetrahedrons obtained by dividing $\Omega$ into equally sized cubes, which provides us with a background mesh. Subsequently each cube of the background mesh is divided into twelve identical tetrahedrons, see also Fig.(5.1). In table 5.1, specifications of the different meshes are given. In all simulations, except for case II


FIgure 5.1: Representation of computational domain.
(see table 5.1), the computational domain has dimensions $2 L_{x} \times 2 L_{y} \times 2 L_{z}=200 \times 200 \times 10$. The background mesh dimensions are given by $N_{x}, N_{y}$ and $N_{z}$, while $n_{p}$ denotes the number of processors used in the computation. For $n_{p}=1$ the non-parallel code was used for the calculations. $n_{p}=2-32$ means that simulations where performed on the specified grid on $2,4,8,16$ and 32 processors. $h$ denotes a characteristic mesh-size, and is given by $h=\frac{1}{N_{x}}$.

The output-files written during the computations are tailored to be used in Tecplot, which implies that the results are given at the nodes of the elements, rather than in the centroids (the

| Case | $N_{x}$ | $N_{y}$ | $N_{z}$ | Tetrahedrons | faces $z=0$ | $h$ | $n_{p}$ |
| :--- | :---: | :---: | :---: | ---: | ---: | ---: | :--- |
| IIa | 40 | 40 | 2 | 38,400 | 3,200 | $\frac{1}{40}$ | $1,2-32$ |
| IIb | 40 | 40 | 10 | 192,000 | 3,200 | $\frac{1}{40}$ | 1 |
| III | 80 | 80 | 4 | 307,200 | 12,800 | $\frac{1}{80}$ | 1 |
| IV | 100 | 100 | 5 | 600,000 | $20,000^{\dagger}$ | $\frac{1}{100}$ | $1,2-128$ |
| V | 120 | 120 | 6 | $1,036,800$ | 28,800 | $\frac{1}{120}$ | 64 |
| VI | 160 | 160 | 8 | $2,457,600$ | 51,200 | $\frac{1}{160}$ | 128 |

Table 5.1: Mesh specifi cations and number of processors used for the computations for the different cases
discontinuous Galerkin method is a cell centered method). When the unknown coefficients in the discontinuous Galerkin method have been evaluated at a certain time-step, the solution in each point of an element can be reconstructed. This means that for a node in the mesh which is common to, say, six elements there are six solutions for that node. The node values are obtained by averaging the node values of all the elements which have the node in common. When we look at the results in the subsequent subsections, we are not just looking at the result of the Discontinuous Galerkin method, but the result of both the Discontinuous Galerkin method and the averaging.

### 5.4.1 Verifi cation and accuracy

Figs.(5.2) to (5.7) show contour-plots for the velocity perturbations, $u^{\prime}$ and $v^{\prime}$, and pressure perturbation $p^{\prime}$ obtained for case VI, at $t=20$ and $t=40$. In all six figures we used the same contour levels. In the figures the solid lines show the plot levels for which the variables have positive values. The dotted lines represent negative values. The maximum and minimum presented contour levels are, respectively, 0.01 and -0.01 . In between the minimum and maximum the contour-levels are separated by a 0.0025 increment.

The wavefronts in Figs.(5.2) to (5.7) appear smooth. Obviously the region with perturbations for $t=40$ is larger than that for $t=20$ and has convected further away from where it originated $(x, y)=(0,0)$. Furthermore, the front of the wave clearly appears to move further away from its convected center.

The results presented in Figs.(5.2) to (5.7), obtained on the finest mesh of case VI, look nicely symmetric as they should. In Fig.(5.8) the result for the pressure perturbation for case IV is shown at $t=20$. The pressure result for case IV is (somewhat) less smooth than the result for case VI, presented in Fig.(5.6). The figure is used to define three lines $L_{1}, L_{2}$ and $L_{3}$ along which we will present results for case IV. In the figure the coordinate $r$ is furthermore shown.

Fig.(5.9) shows the results obtained for the pressure perturbation for case IV along the lines $L_{1}, L_{2}$ and $L_{3}$ for $t=20$. Since the exact solution is axi symmetric, the results along the three lines should, ideally, be identical. From Fig.(5.9) it is observed that the results along $L_{1}$ and $L_{2}$ hardly differ. However, along the line $L_{3}$ we observe a lower resolution of the peaks due to the smaller number of grid points along this line. Apart from near these peaks, the numerical results appear symmetrical.

Fig.(5.10) presents the results obtained for $p^{\prime}$, along the line $x=10$ for $t=20$, for the cases V and VI as well as the analytical solution. The location of the disturbance is accurately resolved in both cases. The zoom-in of the region $-30<r<-10$, presented in


Figure 5.2: Contour-plot for the velocity perturbation in $x$-direction $u^{\prime}$ obtained for case VI at time $t=20$.


Figure 5.4: Contour-plot for the velocity perturbation in $y$-direction $v^{\prime}$ obtained for case VI at time $t=20$.


Figure 5.3: Contour-plot for the velocity perturbation in $x$-direction $u^{\prime}$ obtained for case VI at time $t=40$.


Figure 5.5: Contour-plot for the velocity perturbation in $y$-direction $v^{\prime}$ obtained for case VI at time $t=40$.

Fig.(5.11), shows that the result obtained for case VI shows a slightly better comparison with the analytical solution, than the result obtained for case V , as one might have expected.

Fig.(5.12) presents the results obtained for $p^{\prime}$, along the line $x=20$ for $t=40$, for the cases IV and VI as well as the analytical solution. The results obtained for case VI show a better agreement with the analytical solution than those of case IV. In Figures (5.10) and (5.12) the graph of the analytical solution is obtained by approximating the integrals in Eq.(5.25) by means of a composite Simpson's rule. The number of quadrature points is chosen sufficiently high, so that further increasing the number of quadrature points will not be visible in the figures.


Figure 5.6: Contour-plot for the pressure perturbation $p^{\prime}$ obtained for case VI at time $t=20$.


Figure 5.7: Contour-plot for the pressure perturbation $p^{\prime}$ obtained for case VI at time $t=40$.


FIGURE 5.8: Pressure perturbation for case IV at $t=20$. Definitions of the lines $L_{1}, L_{2}$ and $L_{3}$ and the coordinate $r$.

For $r=0$ the pressure perturbation can be evaluated exactly, see Eq.(5.29). For the pressure perturbation at the convected center we obtain the analytical solution $p^{\prime}(0,20)=$ -0.016624864 . We denote the numerical approximation of $p^{\prime}(0,20)$, computed on a mesh


Figure 5.9: Results for the pressure perturbation for case IV at $t=20$ along the lines $L_{1}, L_{2}$ and $L_{3}$.
with characteristic size $h$, by $\tilde{p}_{h}^{\prime}$. In Fig.(5.14.a) we have plotted $\epsilon=\left|\tilde{p}_{h}^{\prime}-p^{\prime}(0,20)\right|$ vs. $h^{-1}$ on a double logarithmic scale. The values of the characteristic mesh size $h$ related to the various cases are presented in table (5.1). The results of case III, V and VI can be connected by a line which is nearly straight. The result obtained on the coarse mesh of case II only deviates a little from this line. (Note that the result obtained for case IV is not taken into consideration, since the result is not measured in the plane $z=0$.) The slope of the line gives us the order of the method in the point $(r, t)=(0,20)$. The slope suggests $5^{t h}$-order accuracy $\left(h^{2 p+3}\right)$ in this specific point, $(r, t)=(0,20)$. We do know from the 1D wave-propagation analysis, presented in chapter 4, that super-convergence in discontinuous Galerkin methods is possible. However, occurence of super-convergence have only been reported in 1D. In section 4.3.4 it was observed that dispersion and dissipation errors of the most accurate eigenvalue decays at order $2 p+3$ and $2 p+2$, respectively. Recently, Cockburn et al. [?] reported that it is possible to obtain an approximation of order $2 p+1$ when employing polynomials of degree $\leq p$. This still does not explain why we observe a decay of the error like $h^{2 p+3}$ here, except that around the point $(r, t)=(0,20)$ the solution is very smooth.

Assuming that our method is $5^{t h}$-order accurate in $(r, t)=(0,20)$, we can apply Richardson extrapolation ([64]) to obtain a prediction of the exact solution. Employing Richardson extrapolation we assume that the following holds:

$$
\begin{equation*}
\tilde{p}_{h}^{\prime}=a+b h^{5} . \tag{5.30}
\end{equation*}
$$

Using the numerical results of cases V and VI we can obtain the coefficients $a$ and $b$. Coefficient $a$ gives the prediction of the exact solution $p^{\prime}(0,20)$, we obtain $a=-0.016616442$. The relative error $\left|\frac{a-p^{\prime}(0,20)}{p^{\prime}(0,20)}\right|$ is approximately $0.05 \%$. Fig.(5.15) shows the polynomial of Eq.(5.30) together with the numerical results of the different cases.

Above we have considered the accuracy of the numerical solution for the pressure in the specific and special point $r=0$. In the remainder of this section we are looking for an ap-


Figure 5.10: Comparison of numerical results, obtained for case V and VI along the line $x=10$ for $t=20$, with the analytical solution $p^{\prime}(r, 20)$.


Figure 5.11: Zoom-in of the region $-30<r<-10$.
propriate measure for the accuracy of the numerical solution throughout the computational domain. Would the expansion coefficients of the approximate solution (see Eq.(3.15) in chap-


Figure 5.12: Comparison of numerical results, obtained for case IV and VI along the line $x=20$ for $t=40$, with the analytical solution $p^{\prime}(r, 40)$.


Figure 5.13: Magnification of the region $-50<r<$ -30 .
ter 3) be available during or after computation, the norm

$$
\begin{equation*}
\left(\sum_{j} \int_{\Omega_{j}}\left|u-u_{h}\right| d \Omega\right)^{1 / 2} \tag{5.31}
\end{equation*}
$$



FIGURE 5.14: Grid convergence study for the pressure perturbation in $(r, t)=(0,20)$.


Figure 5.15: Fifth-order Richardson extrapolation polynomial through the numerical results of case V and VI.
could be evaluated. However, as previously mentioned, the computational results are written to file in the node points of the grid. Alternatively, the following three different situations, for which a norm will be evaluated, will be considered:
situation 1: Consider the grid points common t the grids of case III, V and VI. $j$
$\{1, n\}$ is the index over these grid points, $n=5043$. Let the sequence at which the points are visited be fixed and given.
situation 2: Consider the grid points common to the grids of case III, IV, V and VI. $j \in\{1, n\}$ is the index over these grid points, $n=882$. Let the sequence at which the points are visited be fixed and given.
situation 3: Consider the grid points in $z=0$ common to the grids of case III, V and VI. $j \in\{1, n\}$ is the index over these grid points, $n=1681$. Let the sequence at which the points are visited be fixed and given.

Definition $\mathcal{N}_{n}^{\infty}$
Let $\mathcal{X}_{n}$ be a Banach space ${ }^{2}$ of $n$-vectors, and let $\Delta u^{n} \in \mathcal{X}_{n} . \Delta u^{n}=\left(\Delta u_{1}^{n}, \Delta u_{2}^{n}, \ldots, \Delta u_{n}^{n}\right)^{T}$. Let us define the norm:

$$
\begin{equation*}
\mathcal{N}_{n}^{\infty}\left(\Delta u^{n}\right) \equiv\left\|\Delta u^{n}\right\|=\sup _{j \in\{1, n\}}\left|\Delta u_{j}^{n}\right| \tag{5.32}
\end{equation*}
$$

where $\Delta u=u_{h}-u_{\text {exact }}, u_{\text {exact }}$ is the exact solution and $u_{h}$ is the numerical result. Next we will obtain the norm for the pressure for the three situations described above. For the analytic solution, the integral in Eq.(5.25) can be evaluated employing numeric quadrature to any desired order of accuracy. The results for $\mathcal{N}_{n}^{\infty}\left(\Delta \tilde{u}^{n}\right)$ for the pressure are presented in Figs.(5.16), (5.17) and (5.18).


Figure 5.16: The norm $\mathcal{N}_{n}^{\infty}\left(\Delta \tilde{u}^{n}\right)$ for situation 1 for the pressure at $t=20 \mathrm{vs} . h^{-1}$, presented on a doublelogarithmic scale.

[^4]

Figure 5.17: The norm $\mathcal{N}_{n}^{\infty}\left(\Delta \tilde{u}^{n}\right)$ for situation 2 for the pressure at $t=20 \mathrm{vs} . h^{-1}$, presented on a doublelogarithmic scale.


Figure 5.18: The norm $\mathcal{N}_{n}^{\infty}\left(\Delta \tilde{u}^{n}\right)$ for situation 3 for the pressure at $t=20 \mathrm{vs} . h^{-1}$, presented on a doublelogarithmic scale.

Johnson and Pitkärata ([45]) prove that when the basis functions are polynomials of degree
$p$, the order of accuracy is at least $p+\frac{1}{2}$. In most practical cases [6], however, the order of accuracy is observed to be $p+1$. We used $p=1$ for this verification case, from which we expect the order of accuracy of the numerical results to be between $1 \frac{1}{2}$ and 2 . From Figs.(5.16) to (5.18) it is observed that the obtained accuracy, measured by the norm $\mathcal{N}_{n}^{\infty}$, is at least $2 \frac{1}{2}$.

### 5.4.2 Speed-up

A performance test has been carried out for case IIa and IV on TERAS, which is a 1024CPU platform ([78]) consisting of two 512-CPU SGI Origin 3800 systems. It has a peak performance of 1 TFlops ( $10^{12}$ floating point operations per second), it is fitted with 500 Mhz R14000 CPU's organized in 256 4-CPU nodes and possesses 1 TByte of total memory. The speed-up is measured in terms of the ratio of the user CPU-times, where the two processor-job serves as reference.


Figure 5.19: Speed-up measured for case IIa (100 time steps) and case IV (1000 time steps) on Origin 3800. Result for case IIa and IV for up to 32 processors.

From Fig.(5.19) it can be seen that near-linear speed-up is obtained for case IV with about 0.6 million tetrahedrons. Slightly superlinear speed-up is obtained on 4 processors for both case IIa and IV, which is probably caused by a more efficient cache performance. Fig.(5.19) shows furthermore that by dividing the domain of case IIa over more than 8 processors, the number of elements assigned to each processor becomes too small (from an efficiency point of view) and the communication overhead becomes apparent. From Fig.(5.20) we observe that for case IV we have a near-linear speed-up, up to 64 processors. On 128 processors the relative speed-up (relative to $n_{p}=2$ ) has dropped to $60 \%$.

On 128 processors approximately $8.10^{9}$ floating point operations were performed per sec-


Figure 5.20: Speed-up measured for case IV (1000 time steps) on Origin 3800 for up to 128 processors.
ond (8 Gflops). For the performance test for case IV, the computation involved 1000 time steps. The computation took less than 8 minutes on 128 processors (elapsed time). TERAS has a peak-performance of 1 Gflops per processor, for case IV the code was observed to operate, on average, at $10 \%$ of this peak when up to 8 processors were used. Using increasingly more processors (more than 8 ) resulted in a gradual drop in performance to approximately $6 \%$ of the peak-performance, when 128 processors were used, which is thought to be due to the increased communication overhead.

# ACOUSTIC RADIATION FROM VIBRATING WALL 

## 6.1 introduction

In this chapter the problem of acoustic radiation from a vibrating wall segment inside an infinite rectangular duct is considered. The objective of this chapter is to compare numerical and analytical results in order to verify the numerical algorithm.

The choice for a problem involving a rectangular duct was made because of practical considerations. In the preceding chapter, dealing with the convection of a two-dimensional Gaussian pulse into (2D-) free space, the required number of elements was observed to be relatively high. The number of elements needed was chosen so high in order to enable a thorough comparison of the numerical results with the analytical solution. For the confined space within a rectangular duct the number of tetrahedral elements is much smaller than for a radiation problem into three (or two-) dimensional free space.

The vibrating wall problem in a duct was studied by Kuijpers, Rienstra, Verbeek and Verheij [50]. In Kuijpers et al. [50], a mathematical model is developed for the acoustic radiation in baffled cylindrical ducts of finite length. The model was used for the design process of MRI (Magnetic Resonance Imaging) scanners. The mathematical procedure presented in Kuijpers et al. [50] combines the techniques of separation of variables, Fourier transformation and Fourier series expansion. The procedure is used here to solve a similar problem in a rectangular duct. In many of the well-known textbooks, e.g. Crighton [27], Morse \& Ingard [59] and Pierce [63], but also Jones [46] and Morse \& Feshbach [60], the subject of propagation and diffraction of sound waves in ducts (usually cylindrical ducts) is covered. In the textbook by Pierce [63] one chapter is devoted to the problem of radiation from vibrating bodies and another to the radiation from sources near and on solid surfaces. Filippi et al. [32] and Morse \& Feshbach [60] describe the problem of a vibrating piston, located at one of the ends of the duct, which generates sound in the duct. Also radiation from (point) sources in ducts is described in literature, however, the vibration of a finite part of the wall that radiates sound into the rectangular duct has, to the author's knowledge, not been covered as such. The mentioned textbooks have, however, been of great importance for the understanding of the problem.

In Kuijpers et al. [50] the solution is provided for the wall vibration problem inside a cylindrical duct. The cylinder is of finite length, the mean flow is assumed absent and the solution is presented in the frequency domain. In this chapter the analytical solution will be given in the time domain, for both the situation without mean flow and the situation in which the mean flow is assumed uniform. Furthermore, the numerical solution is presented for the
situation in which the mean flow is a Hagen-Pouseuille flow.
The vibrating wall problem is not an aeroacoustical problem, because the acoustic source is not of aerodynamic nature. However, the numerical algorithm has been developed to be applied for propagation problems and not for the identification of sources of sound. Therefore thevibrating-wall problem at hand is well-suited as verification problem for the numerical algorithm.

We continue this chapter by describing the vibrating-wall problem in some detail and by introducing the used notation in section 6.2. In section 6.3 the well-known duct modes are presented. The duct modes are obtained from solving the wave equation in the interior of the duct together with hard (rigid) wall boundary conditions on all bounding surfaces. These duct modes will later prove to be the building blocks for the vibrating-wall solution. In section 6.4 the analytical solution for the vibrating wall problem in the duct without mean flow will be presented, with the wall vibration modelled through a normal velocity boundary condition (time derivative of wall displacement). The normal velocity can be represented by a general function of the two wall coordinates. In section 6.4.1 and 6.4.2 the analytical solution is presented for two specific choices of the normal velocity function.

Subsequently, in section 6.5 the problem is extended by introducing a uniform mean flow. Additionally the analytical solution for the two specific choices for the normal velocity function are presented for the case with uniform mean flow.

In chapter 7 the analytical and numerical results are compared.

### 6.2 Problem description

Consider an infinite rectangular duct of height $h$ and width $b$, see Fig.(6.1). In the middle of the duct a finite part of the duct bottom wall, of length $2 l$, is allowed to vibrate. In this chapter we consider the sound field generated by this vibrating wall segment inside the infinite duct. It is assumed that the problem can be described by the linearized Euler equations, which have been presented in chapter 2. In section 2.5 of that chapter it has been presented how the linearized Euler equations are written in dimensionless form. The scaling parameters which are used in this chapter to obtain dimensionless equations are: $h, \rho_{r e f}, c_{r e f}$, where $\rho_{r e f}$ and $c_{r e f}$ represent an appropriate reference density and velocity, respectively. For $c_{r e f}$ we use the ambient speed of sound of the mean flow, $c_{r e f}=c_{0}$. It is noted that in this chapter the notation differs slightly from that of chapter 2 . As in chapter 2 , the aeroacoustic perturbations $\left(\rho^{\prime}, u_{i}^{\prime}, p^{\prime}\right)$ are assumed to be small compared to the mean flow properties $\left(\rho_{0}, u_{i 0}, p_{0}\right)$, i.e. Eq.(2.47):

$$
\left|q^{\prime}\right|=\left|q_{0}\right| \mathcal{O}(\epsilon), \quad \epsilon \ll 1
$$

where $q$ is either $\rho, u_{i}$ or $p$. In the current chapter the perturbation density, velocities and pressure are rescaled:

$$
\begin{equation*}
q=\frac{q^{\prime}}{\epsilon} . \tag{6.1}
\end{equation*}
$$

and we solve for $q$ rather than for $q^{\prime}$. Because both the equations and the boundary conditions are linear, the scaling will not influence the solution.

Note that the origin of the Cartesian coordinate system is in the mid-section of the duct at one of the lower corners. In the origin we define the orthogonal unit coordinate vectors $\mathbf{e}_{x}, \mathbf{e}_{y}$


Figure 6.1: Infi nite rectangular duct with vibrating wall segment.
and $\mathbf{e}_{z}$.
For the mean flow or background flow velocities, expressed in terms of Mach numbers, we assume $\mathbf{u}_{0}=(M, 0,0)^{T}$. Under this assumption the dimensionless acoustic density perturbation $\rho$, velocity field $\mathbf{u}=(u, v, w)^{T}$ and pressure perturbation $p$ satisfy the following linearized continuity equation, linearized momentum equation and linearized energy equation (isentropic flow):

$$
\begin{align*}
& \frac{\partial \rho}{\partial t}+M \frac{\partial \rho}{\partial x}+\frac{\partial u_{j}}{\partial x_{j}}=0  \tag{6.2}\\
& \frac{\partial u_{i}}{\partial t}+M \frac{\partial u_{i}}{\partial x}+\frac{\partial p}{\partial x_{i}}=0 \tag{6.3}
\end{align*}
$$

and

$$
\begin{equation*}
p=\rho, \tag{6.4}
\end{equation*}
$$

respectively.
The wall can be described by the position vector $\mathbf{x}_{s}$ on the moving surface $(S)$ :

$$
\begin{equation*}
\mathbf{x}_{s}(x, y, t)=x \mathbf{e}_{x}+y \mathbf{e}_{y}+z_{s}(x, y, t) \mathbf{e}_{z} \tag{6.5}
\end{equation*}
$$

The total velocity vector $\mathbf{U}$ is given by:

$$
\begin{equation*}
\mathbf{U}(x, y, z, t)=\left(M+u^{\prime}\right) \mathbf{e}_{x}+v^{\prime} \mathbf{e}_{y}+w^{\prime} \mathbf{e}_{z} . \tag{6.6}
\end{equation*}
$$

Since the surface is assumed to be impenatrable, no flow may pass through the wall. This can be imposed by means of the kinematic condition $\frac{d F}{d t}=0$, where $F(x, y, z, t) \equiv z_{s}(x, y, t)-z$ is the definition of the moving surface:

$$
\begin{align*}
\left(\frac{d F}{d t}\right)_{s}=0 \quad & \Rightarrow\left[\frac{\partial F}{\partial t}+\nabla F \cdot \frac{d \mathbf{x}_{s}}{d t}\right]_{s}=0 \\
& \Rightarrow\left[\frac{\partial F}{\partial t}+\nabla F \cdot \mathbf{U}\right]_{s}=0 \tag{6.7}
\end{align*}
$$

With

$$
\begin{equation*}
\frac{\partial F}{\partial t}=\frac{\partial z_{s}}{\partial t}, \quad \nabla F=\frac{\partial z_{s}}{\partial x} \mathbf{e}_{x}+\frac{\partial z_{s}}{\partial y} \mathbf{e}_{y}-\mathbf{e}_{z} \tag{6.8}
\end{equation*}
$$

we obtain:

$$
\begin{equation*}
\left[\frac{\partial z_{s}}{\partial t}+\left(M+u^{\prime}\right) \frac{\partial z_{s}}{\partial x}+v^{\prime} \frac{\partial z_{s}}{\partial y}-w^{\prime}\right]_{s}=0 \tag{6.9}
\end{equation*}
$$

We assume that the amplitude of the surface displacement is small $(\mathcal{O}(\epsilon))$ compared to the acoustic wave length and the surface dimensions, $2 l \times b$. Based on these assumptions it is consistent to linearize Eq.(6.9) with respect to the stationary reference surface $z=0$ (Pierce [63]). This leads to:

$$
\begin{equation*}
w^{\prime}(x, y, 0, t)=\frac{\partial z_{s}^{\prime}}{\partial t}+M \frac{\partial z_{s}^{\prime}}{\partial x} \tag{6.10}
\end{equation*}
$$

As mentioned above, the perturbation velocity is rescaled in this chapter, i.e. $w=w^{\prime} / \epsilon$. In addition also the small surface displacement will be rescaled $z_{s}=z_{s}^{\prime} / \epsilon$.

In the first three sections of this chapter it is assumed that the mean flow velocity is zero, i.e. quiescent conditions with $M=0$. Under these conditions Eqs.(6.150), (6.151) and (6.152)) become:

$$
\begin{align*}
& \frac{\partial \rho}{\partial t}+\frac{\partial u_{j}}{\partial x_{j}}=0  \tag{6.11}\\
& \frac{\partial u_{i}}{\partial t}+\frac{\partial p}{\partial x_{i}}=0 \tag{6.12}
\end{align*}
$$

and

$$
\begin{equation*}
p=\rho, \tag{6.13}
\end{equation*}
$$

respectively. For $M=0$ the displacement of the surface results in a normal velocity boundary condition (Eq.(6.10)):

$$
\begin{equation*}
u_{n}=-w=-\frac{\partial z_{s}}{\partial t} \tag{6.14}
\end{equation*}
$$

We specify the normal velocity at the boundary $z=0$ by:

$$
\begin{equation*}
u_{n}(x, y, t)=\psi(x, y, t) H(l-|x|), \quad z=0 \tag{6.15}
\end{equation*}
$$

where $H$ is the Heaviside function and $l$ is dimensionless and finite.
Eliminating the velocity and density from Eqs.(6.11) to (6.13), we obtain the wave equation for the pressure, in dimensionless form:

$$
\begin{equation*}
\frac{\partial^{2} p}{\partial t^{2}}-\nabla^{2} p=0, \quad \nabla^{2}=\frac{\partial^{2}}{\partial x^{2}}+\frac{\partial^{2}}{\partial y^{2}}+\frac{\partial^{2}}{\partial z^{2}} \tag{6.16}
\end{equation*}
$$

Taking the inner product of the momentum equation, Eq.(6.12), with the unit normal, $\mathbf{n}_{0}=$ $-\mathbf{e}_{z}$, at $z=0$ we obtain as boundary condition for the pressure:

$$
\begin{equation*}
\left.\frac{\partial p}{\partial z}\right|_{z=0}=\frac{\partial u_{n}}{\partial t}=\frac{\partial \psi}{\partial t} H(l-|x|) \tag{6.17}
\end{equation*}
$$



Figure 6.2: Side view of vibrating wall segment.

On the other walls we have the hard-wall conditions:

$$
\begin{equation*}
\left.\frac{\partial p}{\partial z}\right|_{z=1}=0,\left.\quad \frac{\partial p}{\partial y}\right|_{y=0}=0,\left.\quad \frac{\partial p}{\partial y}\right|_{y=b}=0 \tag{6.18}
\end{equation*}
$$

With hard-walls it is meant completely rigid impenetrable walls. Such walls are also called rigid or solid walls.

In the following sections we will solve Eq.(6.16) with the boundary conditions given by Eq.(6.17) and Eq.(6.18). The solution procedure we use is based on the solution procedure presented by Kuijpers et al. [50] and in Kuijpers [49]. In the solution procedure we make use of Fourier transformations and Fourier series expansions. For a function $f(x, t)$ we define the following Fourier transformations, one in time:

$$
\begin{equation*}
\tilde{f}(x, \omega) \equiv \frac{1}{2 \pi} \int_{-\infty}^{\infty} f(x, t) e^{-i \omega t} d t \Rightarrow f(x, t)=\int_{-\infty}^{\infty} \tilde{f}(x, \omega) e^{i \omega t} d \omega \tag{6.19}
\end{equation*}
$$

and one in space:

$$
\begin{equation*}
\hat{f}(\xi, \omega) \equiv \frac{1}{2 \pi} \int_{-\infty}^{\infty} \tilde{f}(x, \omega) e^{i \xi x} d x \Rightarrow \tilde{f}(x, \omega)=\int_{-\infty}^{\infty} \hat{f}(\xi, \omega) e^{-i \xi x} d \xi \tag{6.20}
\end{equation*}
$$

### 6.3 Duct modes

In this section it is assumed that $\psi=0$, which implies that $u_{n}$ is zero as well. Eq.(6.17) now describes the hard-wall boundary condition. Fourier transformation in time of Eq.(6.16) results in the Helmholtz equation for $\tilde{p}(x, y, z, \omega)$ :

$$
\begin{equation*}
\nabla^{2} \tilde{p}+\omega^{2} \tilde{p}=0 \tag{6.21}
\end{equation*}
$$

Fourier transformation in time of Eq.(6.17) and Eq.(6.18) results in:

$$
\begin{align*}
& \frac{\partial \tilde{p}}{\partial y}=0, \quad \text { for } y=0, y=b \\
& \frac{\partial \tilde{p}}{\partial z}=0, \quad \text { for } z=0, z=1 \tag{6.22}
\end{align*}
$$

Eq.(6.21) can be solved employing the method of separation of variables. Hereto we substitute $\tilde{p}(x, y, z, \omega)=X_{0}(x, \omega) Y_{0}(y, \omega) Z_{0}(z, \omega)$ into Eq.(6.21) which, after some rearranging, results in:

$$
\begin{equation*}
\frac{1}{X_{0}} \frac{\partial^{2} X_{0}}{\partial x^{2}}+\frac{1}{Y_{0}} \frac{\partial^{2} Y_{0}}{\partial y^{2}}+\frac{1}{Z_{0}} \frac{\partial^{2} Z_{0}}{\partial z^{2}}+\omega^{2}=0 \tag{6.23}
\end{equation*}
$$

Upon introducing the constants $\alpha$ and $\beta$ (e.g. [86]) such that

$$
\begin{align*}
& \frac{1}{Y_{0}} \frac{\partial^{2} Y_{0}}{\partial y^{2}}=-\alpha^{2}  \tag{6.24}\\
& \frac{1}{Z_{0}} \frac{\partial^{2} Z_{0}}{\partial z^{2}}=-\beta^{2} \tag{6.25}
\end{align*}
$$

we obtain for $X(x, \omega)$ the equation:

$$
\begin{equation*}
\frac{1}{X_{0}} \frac{\partial^{2} X_{0}}{\partial x^{2}}+\left(\omega^{2}-\alpha^{2}-\beta^{2}\right)=0 \tag{6.26}
\end{equation*}
$$

Eqs.(6.24), (6.25) and (6.26), together with the boundary conditions, can easily be solved to obtain the well known duct modes in the frequency domain (see also, amongst others, Rienstra and Hirschberg [70]):

$$
\begin{equation*}
\tilde{p}(x, y, z, \omega)=\sum_{m=0}^{\infty} \sum_{n=0}^{\infty} \cos \left(\alpha_{m} y\right) \cos \left(\beta_{n} z\right)\left(A_{m n}^{0} e^{-i \gamma_{m n} x}+B_{m n}^{0} e^{i \gamma_{m n} x}\right) \tag{6.27}
\end{equation*}
$$

where

$$
\begin{equation*}
\alpha_{m}=\frac{m \pi}{b}, \quad \beta_{n}=n \pi, \quad \gamma_{m n}=\sqrt{\omega^{2}-\alpha_{m}^{2}-\beta_{n}^{2}}, \quad n \in N \tag{6.28}
\end{equation*}
$$

and where we $\operatorname{Re}\left(\gamma_{m n}\right) \geq 0$ and $\operatorname{Im}\left(\gamma_{m n}\right) \leq 0$ (the solution at $x= \pm \infty$ has to remain bounded). The terms with $e^{-i \gamma_{m n} x}$ represent waves propagating in the positive $x$-direction and terms with $e^{i \gamma_{m n} x}$ represent waves propagating in the negative $x$-direction. In Eq.(6.27) $A_{m n}^{0}$ and $B_{m n}^{0}$ represent the modal amplitudes of the $m n^{t h}$ - right and left running wave, respectively. It is known from literature, e.g. [46], [59], [63] [70], that the duct modes presented in Eq.(6.27) mathematically form a complete set of functions suitable for the construction of any sound field in a duct. The solution in the time domain is obtained by the inverse Fourier transform:

$$
\begin{equation*}
p(x, y, z, t)=\int_{-\infty}^{\infty} \tilde{p}(x, y, z, \omega) e^{i \omega t} d \omega \tag{6.29}
\end{equation*}
$$

### 6.4 Acoustic radiation from vibrating wall segment

Consider now the sound field generated in the rigid walled infinite duct by a vibrating plate of length $2 l$ in the bottom of the duct $(z=0)$. Hereto the Helmholtz equation (Eq.(6.21)) with boundary conditions given by:

$$
\begin{array}{ll}
\frac{\partial \tilde{p}}{\partial y}=0, & \text { for } y=0, y=b, \\
\frac{\partial \tilde{p}}{\partial z}=i \omega \tilde{u}_{n}(x, y, \omega), & \text { for } z=0, \\
\frac{\partial \tilde{p}}{\partial z}=0, & \text { for } z=1, \tag{6.32}
\end{array}
$$

is to be solved for $\tilde{p}(x, y, z, \omega)$.
The solution can be obtained conveniently employing the Fourier transformation in $x$. From Eq.(6.20) we obtain for the pressure the transformation pair:

$$
\begin{align*}
& \hat{p}(\xi, y, z, \omega)=\frac{1}{2 \pi} \int_{-\infty}^{\infty} \tilde{p}(x, y, z, \omega) e^{i \xi x} d x \\
& \tilde{p}(x, y, z, \omega)=\int_{-\infty}^{\infty} \hat{p}(\xi, y, z, \omega) e^{-i \xi x} d \xi \tag{6.33}
\end{align*}
$$

Fourier transformation in $x$ of the time-Fourier-transformed normal velocity results in:

$$
\begin{align*}
\hat{u}_{n}(\xi, y, \omega) & =\frac{1}{2 \pi} \int_{-\infty}^{\infty} \tilde{\psi}(x, y, \omega) e^{i \xi x} H(l-|x|) d x \\
& =\frac{1}{2 \pi} \int_{-l}^{l} \tilde{\psi}(x, y, \omega) e^{i \xi x} d x \tag{6.34}
\end{align*}
$$

In addition, we will expand $\hat{p}$ and $\hat{u}_{n}$ in terms of the modes $\cos \left(\alpha_{m} y\right), \alpha_{m}=\frac{m \pi}{b}$ (see also the previous section), employing a, so called, half range series expansion (e.g. [86]):

$$
\begin{align*}
& \hat{u}(\xi, y, \omega)=\sum_{m=0}^{\infty} \hat{u}_{n_{m}}(\xi, \omega) \cos \left(\alpha_{m} y\right),  \tag{6.35}\\
& \hat{u}_{n_{m}}(\xi, \omega)= \begin{cases}\frac{2}{b} \int_{0}^{b} \hat{u}(\xi, y, \omega) \cos \left(\alpha_{m} y\right) d y, & m \neq 0 \\
\frac{1}{b} \int_{0}^{b} \hat{u}(\xi, y, \omega) d y, & m=0\end{cases} \tag{6.36}
\end{align*}
$$

For the pressure we have:

$$
\begin{equation*}
\hat{p}(\xi, y, z, \omega)=\sum_{m=0}^{\infty} \hat{p}_{m}(\xi, z, \omega) \cos \left(\alpha_{m} y\right) \tag{6.37}
\end{equation*}
$$

The pressure given by Eq.(6.37) now satisfies the boundary conditions given by Eq.(6.30). Fourier transforming in $x$, the Helmholtz equation becomes:

$$
\begin{equation*}
\frac{\partial^{2} \hat{p}}{\partial y^{2}}+\frac{\partial^{2} \hat{p}}{\partial z^{2}}+\left(\omega^{2}-\xi^{2}\right) \hat{p}=0 \tag{6.38}
\end{equation*}
$$

Upon substitution of Eq.(6.37) into Eq.(6.38) we obtain:

$$
\begin{equation*}
\sum_{m=0}^{\infty} \cos \left(\alpha_{m} y\right)\left\{\frac{\partial^{2} \hat{p}_{m}}{\partial z^{2}}+\left(\omega^{2}-\xi^{2}-\alpha_{m}^{2}\right)\right\}=0 \tag{6.39}
\end{equation*}
$$

Hence, every $\hat{p}_{m}$ has to satisfy Eq.(6.39) and the boundary conditions given by Eq.(6.31) and Eq.(6.32). We obtain for $\hat{p}_{m}$ :

$$
\begin{equation*}
\hat{p}_{m}(\xi, z, \omega)=i \omega \hat{u}_{n_{m}}(\xi, \omega) \frac{\cos \left((z-1) \gamma_{m}\right)}{\gamma_{m} \sin \left(\gamma_{m}\right)} \tag{6.40}
\end{equation*}
$$

from which we have, employing Eq.(6.37), for $\hat{p}$ :

$$
\begin{equation*}
\hat{p}(\xi, y, z, \omega)=i \omega \sum_{m=0}^{\infty} \cos \left(\alpha_{m} y\right) \hat{u}_{n_{m}}(\xi, \omega) \frac{\cos \left((z-1) \gamma_{m}\right)}{\gamma_{m} \sin \left(\gamma_{m}\right)} . \tag{6.41}
\end{equation*}
$$

Employing the inverse Fourier transform from $\xi$ to $x$ (Eq.(6.33)), we obtain for the pressure in the frequency domain:

$$
\begin{equation*}
\tilde{p}(x, y, z, \omega)=i \omega \sum_{m=0}^{\infty} \cos \left(\alpha_{m} y\right) \int_{-\infty}^{\infty} \hat{u}_{n_{m}}(\xi, \omega) \frac{\cos \left((z-1) \gamma_{m}\right)}{\gamma_{m} \sin \left(\gamma_{m}\right)} e^{-i \xi x} d \xi \tag{6.42}
\end{equation*}
$$

This last expression can be conveniently rewritten by transforming $\hat{u}_{n_{m}}$ backwards from $\xi$ to $x$ (Eq.(6.34)):

$$
\begin{equation*}
\tilde{p}(x, y, z, \omega)=\frac{i \omega}{2 \pi} \sum_{m=0}^{\infty} \cos \left(\alpha_{m} y\right) \int_{-l}^{l} \int_{-\infty}^{\infty} \frac{\cos \left((z-1) \gamma_{m}\right)}{\gamma_{m} \sin \left(\gamma_{m}\right)} e^{-i\left(x-x^{\prime}\right) \xi} d \xi \tilde{\psi}_{m}\left(x^{\prime}, \omega\right) d x^{\prime} \tag{6.43}
\end{equation*}
$$

where

$$
\begin{align*}
& \tilde{\psi}(x, y, \omega)=\sum_{m=0}^{\infty} \tilde{\psi}_{m}(x, \omega) \cos \left(\alpha_{m} y\right)  \tag{6.44}\\
& \tilde{\psi}_{m}(x, \omega)= \begin{cases}\frac{2}{b} \int_{0}^{b} \tilde{\psi}(x, y, \omega) \cos \left(\alpha_{m} y\right) d y, & m \neq 0 \\
\frac{1}{b} \int_{0}^{b} \tilde{\psi}(x, y, \omega) d y, & m=0\end{cases} \tag{6.45}
\end{align*}
$$

The inverse Fourier transform from $\xi$ to $x$, present in Eq.(6.43), can be evaluated in the complex plane, employing the residue integration method. In the remainder of this section the solution procedure for evaluating this inverse Fourier transform is presented.

## Evaluation of inverse Fourier transform

Let us now evaluate the integral:

$$
\begin{align*}
\int_{-\infty}^{\infty} f(\xi) d \xi, \quad \text { with } f(\xi) & =\frac{\cos \left((z-1) \gamma_{m}\right)}{\gamma_{m} \sin \left(\gamma_{m}\right)} e^{-i\left(x-x^{\prime}\right) \xi} \\
& =\frac{\cos \left((z-1) \sqrt{\Omega_{m}^{2}-\xi^{2}}\right)}{\left(\Omega_{m}-\xi\right)\left(\Omega_{m}+\xi\right) \frac{\sin \left(\sqrt{\left.\Omega_{m}^{2}-\xi^{2}\right)}\right.}{\sqrt{\Omega_{m}^{2}-\xi^{2}}}} e^{-i\left(x-x^{\prime}\right) \xi} \tag{6.46}
\end{align*}
$$

where

$$
\begin{equation*}
\gamma_{m}=\sqrt{\omega^{2}-\alpha_{m}^{2}-\xi^{2}}, \quad \Omega_{m}^{2}=\omega^{2}-\alpha_{m}^{2} \tag{6.47}
\end{equation*}
$$

This inverse Fourier transform can be evaluated conveniently in the complex $\zeta$-plane, where $\operatorname{Re}(\zeta) \equiv \xi$, by applying the residue integration method (see appendix C.1). The integrand in Eq.(6.46) is a so-called meromorphic function, i.e. an analytic function in the complex plane except for isolated poles ([26], [49]).

The integrand in Eq.(6.46) has poles at the zeros of $\gamma_{m}$ and $\sin \left(\gamma_{m}\right)$. The zeros $\zeta$ of $\gamma_{m}(\zeta)$ are given by:

$$
\begin{equation*}
\zeta= \pm \Omega_{m} \tag{6.48}
\end{equation*}
$$

The integral of Eq.(6.46) is evaluated for given $\omega \in(-\infty, \infty)$ and $m \in[0, \infty)$, and the poles $\Omega_{m}$ can be either real or imaginary. Without loss of generality, we will assume here that the poles $\pm \Omega_{m}$ are real. From Eq.(6.46) it can be seen that these poles are of order one, thus so-called simple poles. The zeros $\zeta$ of $\sin \left(\gamma_{m}(\zeta)\right)$ are:

$$
\begin{equation*}
\zeta= \pm \lambda_{m k}, \quad \text { where } \quad \lambda_{m k}^{2}=\Omega_{m}^{2}-\beta_{k}^{2}, \quad \beta_{k}=k \pi . \tag{6.49}
\end{equation*}
$$

Note that $\lambda_{m 0}=\Omega_{m}$. From the Laurent series expansion it can be shown that these poles are also simple poles. The poles $\pm \lambda_{k m}$ can be either real or imaginary. In Fig.(6.3) not only the poles have been presented in the complex $\zeta$-plane, but also the integration contours used to evaluate the integral of Eq.(6.46). The motivation for the choice of integration contours, including the motivation for indenting the contour along the real axis above or below the poles on the real axis, is given below. In Fig.(6.3) we used:

$$
\begin{array}{ll}
\lambda_{m k} \in \mathbb{R} & \forall k<\mu,  \tag{6.50}\\
\lambda_{m k} \in i \mathbb{R} & \forall k \geq \mu .
\end{array}
$$

In order to evaluate Eq.(6.46) we define two contours in the complex $\zeta$-plane, $C^{ \pm}$:

$$
\begin{equation*}
C^{ \pm}=\{-R<\xi<R\} \cup S^{ \pm}, \tag{6.51}
\end{equation*}
$$

where the semi-circles $S^{ \pm}$are given by:

$$
\begin{equation*}
S^{ \pm}=\left\{\zeta=R e^{\mp i \theta}, 0 \leq \theta \leq \pi\right\} \tag{6.52}
\end{equation*}
$$

For $x-x^{\prime}>0$ in Eq.(6.46) we use contour $C^{+}$and for $x-x^{\prime}<0$ contour $C^{-}$, see also Fig.(6.3). The choice for these contours depends on the sign of $x-x^{\prime}$ and is determined by the requirement that the integrand has to remain bounded when the contours are taken to infinity. It is important to note that, for the moment, the contours are of finite length and enclose a finite number of poles. At a later stage the contours are taken to infinity (by letting $R \rightarrow \infty)$ on the premise of enclosing all poles and proper integral behavior.

The poles on the real axis, which are finite in number, hinder defining the contours. The contours are not allowed to run through any of the poles, therefore we have to allow infinitesimal deformations of the contours, such that the contours avoid running through the poles. Within the region of analyticity these deformations do not change the integral of a meromorphic function ([26]). However, such deformations result in the contours running above or below the singularities, and the results are not the same. Clearly we need additional information in order to be able to define suitable deformations of the contours. The missing


FIGURE 6.3: Poles and integration contours in the complex $\zeta$-plane. In this graphical representation $\lambda_{m k} \in \mathbb{R} \forall k<\mu$ and $\lambda_{m k} \in i \mathbb{R} \forall k \geq \mu$.
information is provided by the causality condition (see appendix C.2). In short, the causality condition states the following; A process $p(t)$ that starts by some cause at some finite time $t=t_{0}$, while it vanishes for $t<t_{0}$, is called causal ([70]). With the definition of the Fourier transformation pair presented in Eq.(6.19), Cauchy's theorem states that, if the corresponding Fourier transform $\tilde{p}(\omega)$ is analytic for $\operatorname{Im}(\omega) \leq 0$, there exists a $t_{0} \in \mathbb{R}$ such that $e^{i \omega t_{0}} \tilde{p}(\omega) \rightarrow 0$ for $\omega \rightarrow-i \infty$ and $p(t)$ is causal and vanishes for $t<t_{0}$. So, the solution we are looking for here, is required to be the analytical continuation in $\omega$ from the lower complex $\omega$ half plane ([49]). If a small negative imaginary part is added to $\omega$, the poles $\zeta=\Omega_{m}$ and $\zeta=\lambda_{m k}$ shift into the lower half plane, and the poles $\zeta=-\Omega_{m}$ and $\zeta=-\lambda_{m k}$ shift into the upper half plane. Therefore the contour of integration should be indented below $\zeta=-\Omega_{m}$ and $-\lambda_{m k}$ and above $\zeta=\Omega_{m}$ and $\lambda_{m k}$, as is shown in Fig.(6.3).

In the remainder of this section we will show the solution procedure assuming $x>x^{\prime}$. The solution for $x<x^{\prime}$ is obtained in a similar way. The (counter clock-wise) contour integral over contour $C^{+}$can be split into an integral along a part of the real axis and an integral over the semi-circle $S^{+}=R e^{-i \theta}, 0 \leq \theta \leq \pi$, see also Fig.(6.3:

$$
\begin{equation*}
\oint_{C^{+}} f(\zeta) d \zeta=\int_{R}^{-R} f(\xi) d \xi+\int_{S^{+}} f(\zeta) d \zeta . \tag{6.53}
\end{equation*}
$$

From the residue theorem we have for the simple poles of the present case:

$$
\begin{equation*}
\oint_{C^{+}} f(\zeta) d \zeta=2 \pi i \sum_{k} \underset{\zeta=}{\substack{\operatorname{Res}}} f(\zeta), \tag{6.54}
\end{equation*}
$$

where $\zeta_{k}$ are the poles of $f(\zeta)$ enclosed by $C^{+}$. The residues for the simple poles can be easily evaluated, see also appendix C.1.1; we obtain:

- for $\zeta=\Omega_{m}$ :

$$
\begin{equation*}
\operatorname{Res}_{\zeta=\Omega_{m}} f(\zeta)=-\frac{e^{-i\left(x-x^{\prime}\right) \Omega_{m}}}{2 \Omega_{m}} \tag{6.55}
\end{equation*}
$$

- for $\zeta=\lambda_{m k}, k \in \mathbb{N} \backslash\{0\}$ :

$$
\begin{equation*}
\operatorname{Res}_{\substack{\zeta=\lambda_{m k} \\ k \neq 0}}^{\operatorname{Ren}} f(\zeta)=-\frac{\cos \left(\beta_{k} z\right) e^{-i\left(x-x^{\prime}\right) \lambda_{m k}}}{\lambda_{m k}} \tag{6.56}
\end{equation*}
$$

Note that formally $k \in \mathbb{Z} \backslash\{0\}$, but since $k$ is only present in even functions with respect to $k$, all possibilities are covered taking $k \in \mathbb{N} \backslash\{0\}$.

Next it will be shown that the contribution of the integral over $S^{+}$is zero. Hereto we apply Jordan's lemma, see appendix C.3. When we can show

$$
\begin{equation*}
\lim _{\zeta \rightarrow \infty} f(\zeta)=0, \quad \zeta \in S^{+} \tag{6.57}
\end{equation*}
$$

according to the lemma of Jordan the contribution of the integral over $S^{+}$is indeed zero. Writing $\gamma_{m}$ as:

$$
\begin{equation*}
\gamma_{m}(\zeta)=i \zeta \sqrt{1-\frac{\Omega_{m}^{2}}{\zeta^{2}}} \tag{6.58}
\end{equation*}
$$

we observe that:

$$
\begin{equation*}
\gamma_{m}(\zeta) \rightarrow i \zeta \quad \text { when }|\zeta| \rightarrow \infty, \quad \zeta \in S^{+} \tag{6.59}
\end{equation*}
$$

and

$$
\begin{equation*}
\lim _{|\zeta| \rightarrow \infty} f(\zeta)=\lim _{|\zeta| \rightarrow \infty} \frac{1}{\zeta} \frac{e^{(1-z) \zeta}+e^{-(1-z) \zeta}}{e^{-\zeta}-e^{\zeta}}, \quad \zeta \in S^{+} \tag{6.60}
\end{equation*}
$$

As mentioned before, the contour $S^{+}$does not run through any of the poles, so we do not have to worry about the zero's of $\gamma_{m} \sin \left(\gamma_{m}\right)$ when taking the contour to infinity. Because $z \in[0,1]$ it is therefore clear that the limit evaluates to zero:

$$
\begin{equation*}
\lim _{|\zeta| \rightarrow \infty} \frac{1}{\zeta} \frac{e^{(1-z) \zeta}+e^{-(1-z) \zeta}}{e^{-\zeta}-e^{\zeta}}=0, \quad \zeta \in S^{+} \tag{6.61}
\end{equation*}
$$

That leaves unanswered the question whether all poles are included when the contour is taken to infinity. Note that the poles on the imaginary axis run into (minus) infinity themselves, at equal distances $(\pi)$ from each other. Furthermore, how does the integral, Eq.(6.46), behave when $|\zeta| \rightarrow \infty, \zeta \in S^{+}$? Papoulis ([61]) shows that this problem can be dealt with by defining a sequence of circular arcs with radii tending to infinity, see also appendix C.1.2. Each arc of the sequence encloses all previous arcs and encloses the next singularity which was previously not enclosed, see also Fig.(3.1). Papoulis ([61]) shows that this procedure will lead to the desired result. However, all on the premise of proper integral behavior at infinity.

In the present case we do not have to rely on such a sequence as described by Papoulis ([61]). If we look at the residues, given by Eq.(6.56), we observe that since:

$$
\begin{equation*}
\lambda_{m k}=\sqrt{\Omega_{m}^{2}-(k \pi)^{2}} \rightarrow-i k \pi \quad \text { when } k \rightarrow \infty \tag{6.62}
\end{equation*}
$$

we have:

$$
\begin{equation*}
\lim _{k \rightarrow \infty}\left\{\operatorname{Res}_{\substack{\zeta=\\ k \neq 0}} f(\zeta)\right\}=+\lim _{k \rightarrow \infty} \frac{\cos \left(\beta_{k} z\right) e^{-\left(x-x^{\prime}\right) k \pi}}{i k \pi}=0, \quad x-x^{\prime}>0 \tag{6.63}
\end{equation*}
$$

Clearly the influence of all poles has been taken into account and the integral does behave regularly when the contour $S^{+}$is taken into infinity. We may now conclude that the contribution of the integral over the semi-circle $S^{+}$in Eq.(6.53) vanishes when $R \rightarrow \infty$, so we have:

$$
\begin{equation*}
\int_{-\infty}^{\infty} f(\xi) d \xi=-\oint_{C^{+}} f(\zeta) d \zeta=-2 \pi i \sum_{k}{\underset{\zeta=}{\operatorname{Res}} \zeta_{k}}^{\operatorname{Res}} f(\zeta) \tag{6.64}
\end{equation*}
$$

Resulting for $x-x^{\prime}>0$ we obtain:

$$
\begin{equation*}
\int_{-\infty}^{\infty} f(\xi) d \xi=2 \pi i\left\{\frac{e^{-i\left(x-x^{\prime}\right) \Omega_{m}}}{2 \Omega_{m}}+\sum_{k=1}^{\infty} \cos \left(\beta_{k} z\right) \frac{e^{-i\left(x-x^{\prime}\right) \lambda_{m k}}}{\lambda_{m k}}\right\} . \tag{6.65}
\end{equation*}
$$

The evaluation of the integral for $x-x^{\prime}<0$ is similar. It can be shown that for $x-x^{\prime}<0$ we obtain the result:

$$
\begin{equation*}
\int_{-\infty}^{\infty} f(\xi) d \xi=2 \pi i\left\{\frac{e^{i\left(x-x^{\prime}\right) \Omega_{m}}}{2 \Omega_{m}}+\sum_{k=1}^{\infty} \cos \left(\beta_{k} z\right) \frac{e^{i\left(x-x^{\prime}\right) \lambda_{m k}}}{\lambda_{m k}}\right\} . \tag{6.66}
\end{equation*}
$$

Upon substitution of these results into Eq.(6.43), the solution for the pressure, in the frequency domain, can finally be written as:

$$
\begin{align*}
\tilde{p}(x, y, z, \omega)=-\omega \sum_{m=0}^{\infty} & \cos \left(\alpha_{m} y\right)\left\{\frac{1}{2 \Omega_{m}} \int_{-l}^{l} \tilde{\psi}_{m}\left(x^{\prime}, \omega\right) e^{-i\left|x-x^{\prime}\right| \Omega_{m}} d x^{\prime}\right. \\
& \left.+\sum_{k=1}^{\infty} \frac{\cos \left(\beta_{k} z\right)}{\lambda_{m k}} \int_{-l}^{l} \tilde{\psi}_{m}\left(x^{\prime}, \omega\right) e^{-i\left|x-x^{\prime}\right| \lambda_{m k}} d x^{\prime}\right\} \tag{6.67}
\end{align*}
$$

where

$$
\begin{equation*}
\lambda_{m k}=\sqrt{\Omega_{m}^{2}-\beta_{k}^{2}}, \quad \Omega_{m}=\sqrt{\omega^{2}-\alpha_{m}^{2}}, \quad \beta_{k}=k \pi, \quad \alpha_{m}=\frac{m \pi}{b} \tag{6.68}
\end{equation*}
$$

The solution for the pressure in the time domain can subsequently be obtained upon applying the inverse Fourier transform Eq.(6.19), to transform from $\omega$ to $t$.

Eq.(6.67) can also be written as:

$$
\begin{align*}
\tilde{p}(x, y, z, \omega)= & \sum_{m=0}^{\infty} \cos \left(\alpha_{m} y\right)\left\{\frac{1}{2}\left[A_{m 0} e^{-i x \lambda_{m 0}}+B_{m 0} e^{i x \lambda_{m 0}}\right]\right. \\
& \left.+\sum_{k=1}^{\infty} \cos \left(\beta_{k} z\right)\left[A_{m k} e^{-i x \lambda_{m k}}+B_{m k} e^{i x \lambda_{m k}}\right]\right\} \tag{6.69}
\end{align*}
$$

where the modal amplitudes of the right and left running modes are given by:

$$
\begin{align*}
A_{m k} & =-\frac{\omega}{\lambda_{m k}} \int_{-l}^{l} \tilde{\psi}_{m}\left(x^{\prime}, \omega\right) e^{i x^{\prime} \lambda_{m k}} d x^{\prime}  \tag{6.70}\\
B_{m k} & =-\frac{\omega}{\lambda_{m k}} \int_{-l}^{l} \tilde{\psi}_{m}\left(x^{\prime}, \omega\right) e^{-i x^{\prime} \lambda_{m k}} d x^{\prime} \tag{6.71}
\end{align*}
$$

respectively. When we look at the solution presented in Eq.(6.69) and compare it with the hard-wall duct modes, presented in Eq.(6.27), section 6.3, we observe that the solution for the vibrating wall is expressed in terms of hard-wall modes. This indicates, as is known from literature, e.g. [46], [59], [63] [70], that the duct modes presented in section 6.3 mathematically form a complete set of functions suitable for the construction of any sound field in a duct.

Additionally, it should be mentioned here that by taking the $z$-derivative of the solution presented in Eq.(6.69) and evaluating the result for $z=0$ we obtain $\left.\frac{\partial \tilde{p}}{\partial z}\right|_{z=0}=0$, which does not satisfy the boundary condition of the moving wall. As explained by Rienstra and Hirschberg in [70] and by Kuijpers and Rienstra in [50] for a similar problem in a circular duct, the infinite series for $\frac{\partial \tilde{p}}{\partial z}$ is not uniformly converging. Pointwise, the value at the wall is not equal to the limit to the wall, while it is only the limit which is physically relevant ([70]). It is not easily verified from Eq.(6.69) that the solution near the bottom wall behaves according to the boundary condition. However, it can easily be shown from Eq.(6.42) that:

$$
\begin{equation*}
\frac{\partial \tilde{p}}{\partial z}(x, y, z, \omega)=-i \omega \sum_{m=0}^{\infty} \cos \left(\alpha_{m} y\right) \int_{-\infty}^{\infty} \hat{u}_{n_{m}}(\xi, \omega) \frac{\gamma_{m} \sin \left((z-1) \gamma_{m}\right)}{\gamma_{m} \sin \left(\gamma_{m}\right)} e^{-i \xi x} d \xi \tag{6.72}
\end{equation*}
$$

and

$$
\begin{align*}
\frac{\partial \tilde{p}}{\partial z}(x, y, 0, \omega) & =i \omega \sum_{m=0}^{\infty} \cos \left(\alpha_{m} y\right) \int_{-\infty}^{\infty} \hat{u}_{n_{m}}(\xi, \omega) e^{-i \xi x} d \xi  \tag{6.73}\\
& =i \omega \sum_{m=0}^{\infty} \cos \left(\alpha_{m} y\right) \tilde{u}_{n_{m}}(x, \omega)=i \omega \tilde{u}_{n}(x, y, \omega) \tag{6.74}
\end{align*}
$$

which is exactly the boundary condition given by Eq.(6.31).
The solution presented in Eq.(6.67) or Eq.(6.69) is still very general. In the following two sections we will present the solution for the pressure in the time domain for two different
specific choices for the function $\psi(x, y, t)$. In the first section, section 6.4.1, for $\psi$ a function of time only is chosen, which greatly simplifies the solution process. In the next section, 6.4.2, the function $\psi$ is also a function of $x$. When $\psi$ is a function of $y$ as well, the solution process is complicated by the appearance of branch points. Such a general $\psi(x, y, t)$ has not been considered in the present study.

### 6.4.1 Case A: plunging wall segment

Let us assume for $\psi$ in Eq.(6.15) the following function:

$$
\begin{equation*}
\psi(t)=\sin \left(\omega_{0} t\right) e^{-a t} H(t), \quad \text { where } a \geq 0 \tag{6.75}
\end{equation*}
$$

and where $H$ is the Heaviside function. In this case the whole plate ( $-l \geq x \leq l$ ) moves up


Figure 6.4: Side view of vibrating wall segment.
and down as a rigid body, see Fig.(6.4) for a sketch of this situation. For $a>0$ in Eq.(6.75), the amplitude of the introduced vibration at the wall is exponentially damped because of the exponential term in Eq.(6.75). From Fig.(6.4) it is clear that the vibrating plate can detach from the duct wall, thus creating a discontinuity in the duct wall at $x= \pm l$. Although this is not a very physical situation, it is one of the simplest cases to consider and the solution is considered illustrative. In the next section we will choose $\psi$ such that discontinuities do not occur in the wall surface.

We will consider the solution for this case at a location $x>l$. Let us start by evaluating $\tilde{\psi}_{m}$. Fourier transformation of Eq.(6.75) from $t$ to $\omega$ can be shown to result in:

$$
\begin{equation*}
\tilde{\psi}(\omega)=\frac{1}{2 \pi} \frac{\omega_{0}}{\left[i\left(\omega-\omega_{0}\right)+a\right]\left[i\left(\omega+\omega_{0}\right)+a\right]} \equiv Q(\omega) . \tag{6.76}
\end{equation*}
$$

From Eq.(6.45) it is readily seen that:

$$
\tilde{\psi}_{m}(\omega)= \begin{cases}Q(\omega), & m=0  \tag{6.77}\\ 0, & m \neq 0\end{cases}
$$

For $m=0$ we have $\Omega_{0}=\omega, \lambda_{0 k}=\sqrt{\omega^{2}-\beta_{k}^{2}}$. With these results we can write for the pressure, Eq.(6.67) or Eq.(6.69), in the frequency domain:

$$
\begin{equation*}
\tilde{p}(x, y, z, \omega)=\tilde{p}(x, z, \omega)=\tilde{p}_{0}(x, \omega)+\sum_{k=1}^{\infty} \cos \left(\beta_{k} z\right) \tilde{p}_{k}(x, \omega) \tag{6.78}
\end{equation*}
$$

where:

$$
\begin{align*}
& \tilde{p}_{0}(x, \omega)=\frac{1}{2} A_{00} e^{-i x \omega}=-\frac{Q(\omega)}{2} \int_{-l}^{l} e^{-i \omega\left(x-x^{\prime}\right)} d x^{\prime},  \tag{6.79}\\
& \tilde{p}_{k}(x, \omega)=A_{0 k} e^{-i x \lambda_{0 k}}=-\frac{\omega Q(\omega)}{\lambda_{0 k}} \int_{-l}^{l} e^{-i \lambda_{0 k}\left(x-x^{\prime}\right)} d x^{\prime} . \tag{6.80}
\end{align*}
$$

The integrals in Eq.(6.79) and Eq.(6.80) can be easily evaluated.
To obtain the solution in the time domain we have to evaluate:

$$
\begin{align*}
& p_{0}(x, t)=-\frac{\omega_{0}}{4 \pi i} \int_{-\infty}^{\infty} \frac{e^{-i(x-l) \omega}-e^{-i(x+l) \omega}}{\omega\left[i\left(\omega-\omega_{0}\right)+a\right]\left[i\left(\omega+\omega_{0}\right)+a\right]} e^{i \omega t} d \omega  \tag{6.81}\\
& p_{k}(x, t)=-\frac{\omega_{0}}{2 \pi i} \int_{-\infty}^{\infty} \frac{\omega\left(e^{-i(x-l) \lambda_{0 k}}-e^{-i(x+l) \lambda_{0 k}}\right)}{\lambda_{0 k}^{2}\left[i\left(\omega-\omega_{0}\right)+a\right]\left[i\left(\omega+\omega_{0}\right)+a\right]} e^{i \omega t} d \omega . \tag{6.82}
\end{align*}
$$

The integrals, which are left to be evaluated in Eqs.(6.81) and (6.82), do not look appealing. However, the integral of Eq.(6.81) can be evaluated without too much difficulty.

## Evaluation of $p_{0}$

In order to obtain $p_{0}$ we have to evaluate two integrals of the form:

$$
\begin{equation*}
\int_{-\infty}^{\infty} f(\omega) d \omega, \quad \text { with } f(\omega)=\frac{-e^{i \omega \tau}}{\omega\left[\omega-\left(\omega_{0}+i a\right)\right]\left[\omega-\left(-\omega_{0}+i a\right)\right]} \tag{6.83}
\end{equation*}
$$

where $\tau$ is either $t-(x-l)$ or $t-(x+l)$. This integral can be evaluated in the complex $\omega$-plane by applying the residue integration method (see appendix C.1).

The integrand has three simple poles, namely for $\omega=0, \omega=\omega_{0}+i a$ and $\omega=-\omega_{0}+i a$. Depending on the sign of $\tau$ we define two contours in the complex plane. For $\tau>0$ the integrand only remains bounded in the upper half-plane when the integration contour is taken to infinity. Therefore we choose for $\tau>0$ the contour, $C^{+}$, in the upper half-plane and for $\tau<0$ the contour, $C^{-}$, in the lower half-plane, as depicted in Fig.(6.5). The contours are made up by a segment on the real axis and a semi-circle:

$$
\begin{equation*}
C^{ \pm}=\{-R<\operatorname{Re}(\omega)<R, \operatorname{Im}(\omega)=0\} \cup S^{ \pm} \tag{6.84}
\end{equation*}
$$

where the semi-circles $S^{ \pm}$are given by:

$$
\begin{equation*}
S^{ \pm}=\left\{\omega=R e^{ \pm i \theta}, 0 \leq \theta \leq \pi\right\} \tag{6.85}
\end{equation*}
$$

The definition of these contours leads to a natural splitting of the contour integrals into:

$$
\begin{equation*}
\int_{C^{ \pm}} f(\omega) d \omega=\int_{-\infty}^{\infty} f(\omega) d \omega+\int_{S^{ \pm}} f(\omega) d \omega . \tag{6.86}
\end{equation*}
$$



FIGURE 6.5: Poles and integration contours in the complex $\omega$-plane.

From the causality condition, see appendix C.2, it follows that all poles, so also the pole for $\omega=0$, have to be enclosed by contour $C^{+}$. This is also true when $a=0$ (non-decaying wall vibrations). Because the contour $C^{-}$does not enclose any poles we simply have from Cauchy' theorem (a.o. [26], [27], [64], [71]):

$$
\begin{equation*}
\int_{C^{-}} f(\omega) d \omega=0 \tag{6.87}
\end{equation*}
$$

Furthermore:

$$
\begin{equation*}
\int_{C^{+}} f(\omega) d \omega=2 \pi i \sum_{k=1}^{3} \underset{\omega=\omega_{k}}{\operatorname{Res}_{k}} f(\omega), \tag{6.88}
\end{equation*}
$$

where $\omega_{k}$ denote the (three) poles of $f(\omega)$, enclosed by $C^{+}$.
Writing $f(\omega)=f^{*}(\omega) e^{i \omega \tau}$, it is easily shown that on $S^{ \pm}$we have:

$$
\begin{equation*}
\lim _{\omega \rightarrow \infty} f^{*}(\omega)=0, \quad \omega \in S^{ \pm} \tag{6.89}
\end{equation*}
$$

From the lemma of Jordan (appendix C.3) it now follows that:

$$
\begin{equation*}
\int_{S^{ \pm}} f(\omega) d \omega=0 . \tag{6.90}
\end{equation*}
$$

The residues of the simple poles enclosed by $C^{+}$can be evaluated employing one of the methods presented in appendix C.1.1. It is obtained:

- for $\omega=0$ :

$$
\begin{equation*}
\underset{\omega=0}{\operatorname{Res}} f(\omega)=\frac{1}{\omega_{0}^{2}+a^{2}}, \tag{6.91}
\end{equation*}
$$

- for $\omega=\omega_{0}+i a$ :

$$
\begin{equation*}
\underset{\omega=\omega_{0}+i a}{\operatorname{Res}^{2}} f(\omega)=-\frac{1}{2} \frac{e^{i \omega_{0} \tau} e^{-a \tau}}{\omega_{0}\left(\omega_{0}+i a\right)}, \tag{6.92}
\end{equation*}
$$

- for $\omega=-\omega_{0}+i a$ :

$$
\begin{equation*}
{ }_{\omega=}^{\operatorname{ReS}_{\omega_{0}+i a}} f(\omega)=-\frac{1}{2} \frac{e^{-i \omega_{0} \tau} e^{-a \tau}}{\omega_{0}\left(\omega_{0}-i a\right)} . \tag{6.93}
\end{equation*}
$$

From Eq.(6.86), Eq.(6.88) and Eq.(6.90) we now obtain for the integral:

$$
\begin{equation*}
\int_{-\infty}^{\infty} f(\omega) d \omega=\frac{2 \pi i}{\omega_{0}^{2}+a^{2}}\left\{1-\left[\cos \left(\omega_{0} \tau\right)+\frac{a}{\omega_{0}} \sin \left(\omega_{0} \tau\right)\right] e^{-a \tau}\right\} H(\tau) \tag{6.94}
\end{equation*}
$$

Resulting we obtain for $p_{0}$ :

$$
\begin{equation*}
p_{0}(x, t)=\frac{1}{2} \frac{\omega_{0}}{\omega_{0}^{2}+a^{2}}\left\{g\left(\tau_{1}\right) H\left(\tau_{1}\right)-g\left(\tau_{2}\right) H\left(\tau_{2}\right)\right\}, \quad x>l, \tag{6.95}
\end{equation*}
$$

where

$$
\begin{equation*}
g(\tau)=\left[\cos \left(\omega_{0} \tau\right)+\frac{a}{\omega_{0}} \sin \left(\omega_{0} \tau\right)\right] e^{-a \tau}-1 \tag{6.96}
\end{equation*}
$$

In the above equations, $\tau$ is the time at which the signal, which has been emitted at $t=0$, is observed at location $x$ and time $t$. This time is called the retarded or emission time. The retarded times $\tau_{1}$ and $\tau_{2}$ are given by:

$$
\begin{align*}
& \tau_{1}=t-(x-l)  \tag{6.97}\\
& \tau_{2}=t-(x+l) \tag{6.98}
\end{align*}
$$

Based on the solution presented in Eq.(6.95), an observer, stationed at a certain location $x>l$, would start to observe a signal at $t=x-l$. At this specific point in time the signal, coming from the nearest end of the plate, has covered the distance from the nearest end of the plate to the observer, traveling at the speed of sound $(=1)$. From $t=x+l$ onwards also the contribution from the far end of the plate can be observed.

Clearly $p_{0}$ (Eq.6.95) represents a propagation wave or mode.

## Evaluation of $p_{k}$

Let us now turn to solving $p_{k}$. From Eq.(6.82), we observe that, in order to obtain $p_{k}$, we have to evaluate two integrals of the form:

$$
\begin{equation*}
I(\omega, \bar{x}, t)=\frac{1}{2 \pi i} \int_{-\infty}^{\infty} f(\omega) d \omega, \quad \text { with } f(\omega)=\frac{\omega-e^{i\left(\omega t-\lambda_{0 k} \bar{x}\right)}}{\lambda_{0 k}^{2}\left[\omega-\left(\omega_{0}+i a\right)\right]\left[\omega-\left(-\omega_{0}+i a\right)\right]} \tag{6.99}
\end{equation*}
$$

where

$$
\begin{equation*}
\lambda_{0 k}(\omega)=\sqrt{\omega^{2}-\beta_{k}^{2}}, \tag{6.100}
\end{equation*}
$$

and $\bar{x}=x-l$ or $\bar{x}=x+l$. We have here included the factor $\frac{1}{2 \pi i}$ immediately (unlike in solving $p_{0}$ ). Upon recalling that for $\bar{x}>0$ we have $\operatorname{Re}\left(\lambda_{0 k}\right)>0$ and $\operatorname{Im}\left(\lambda_{0 k}\right)<0$, it follows from Eq.(6.100) that

$$
\begin{array}{ll}
|\omega|<\beta_{k}: & \lambda_{0 k}=-\sqrt{\beta_{k}^{2}-\omega^{2}},  \tag{6.101}\\
|\omega| \geq \beta_{k}: & \lambda_{0 k}=-i \sqrt{\omega^{2}-\beta_{k}^{2}} .
\end{array}
$$

Aparently a large (infinte) number of modes is exponentially decaying (or so-called cut-off) for low frequencies $|\omega|<\beta_{k}$. These specific waves or modes are also known as evanescent waves or decaying modes. For higher frequencies more modes are so-called cut-on; they are propagating. With increasing $k$ an increasing number of modes becomes cut-off. Apparently, $\beta_{k}$ is the so-called cut-off frequency. However, usually the cut-off frequency is the frequency beyond which the modes become cut-off. As we will observe at a later stage it will remain complicated to designate parts of the solution $p_{k}$ as (purely) propagating or evanescent.

Looking at the integrand in Eq.(6.99) we observe that for $\omega= \pm \beta_{k}$ we have branch points, which greatly complicate the evaluation of $p_{k}$. Integrals of the form:

$$
\begin{equation*}
\int_{B} g(\omega) e^{i \omega t-\sqrt{\omega^{2}-k^{2}}} d \omega, \tag{6.102}
\end{equation*}
$$

where $B$ is a suitable path from $-\infty$ to $\infty$ in the complex $\omega$-plane avoiding the branch cuts, are called diffraction integrals by Crighton et al.([27]), because they often appear in diffraction problems.

In order to solve Eq.(6.99) it was found to be beneficial to use (inverse) Laplace transformation instead of (inverse) Fourier transformation. Upon substitution of $\omega=s / i$ into Eq.(6.99) we obtain:

$$
\begin{equation*}
I(s, \bar{x}, t)=\frac{1}{2 \pi i} \int_{-i \infty}^{i \infty} F(s) G(s, \bar{x}) e^{s t} d s \tag{6.103}
\end{equation*}
$$

where:

$$
\begin{align*}
& F(s)=\frac{s}{\left[s^{2}+\beta_{k}^{2}\right]\left[s-\left(i \omega_{0}-a\right)\right]\left[s+\left(i \omega_{0}+a\right)\right]}  \tag{6.104}\\
& G(s, \bar{x})=e^{-\bar{x} \sqrt{s^{2}+\beta_{k}^{2}}} \tag{6.105}
\end{align*}
$$

It is important to note the negative sign in Eq.(6.105). When substituting $\omega=s / i$ into $\lambda_{0 k}$ the argument becomes negative and $\lambda_{0 k}$ becomes purely imaginary. From Fig.(6.3) we observe that for $\bar{x}>0$ we have the branch $\operatorname{Re}\left(\lambda_{0 k}\right)>0$ and $\operatorname{Im}\left(\lambda_{0 k}\right)<0$, and therefore, for $\bar{x}>0$, we obtain:

$$
\begin{equation*}
-i \bar{x} \sqrt{\omega^{2}-\beta_{k}^{2}} \stackrel{\omega=s / i}{=}-i \bar{x}(-i) \sqrt{s^{2}+\beta_{k}^{2}}=-\bar{x} \sqrt{s^{2}+\beta_{k}^{2}} . \tag{6.106}
\end{equation*}
$$

For $\bar{x}<0$ we have to redefine $G(s)$ as $G(s)=e^{+\bar{x} \sqrt{s^{2}+\beta_{k}^{2}}}$. Eq.(6.103) can be solved for both $\bar{x}>0$ and $\bar{x}<0$, however, here we will only show the solution for $\bar{x}>0$.

To solve Eq.(6.103) we can apply the convolution theorem, from which it is obtained:

$$
\begin{equation*}
I(s, \bar{x}, t)=I(\tau, \bar{x}, t)=\int_{0}^{t} F(t-\tau) G(\tau, \bar{x}) d \tau \tag{6.107}
\end{equation*}
$$

Furthermore we will use the Laplace transformation tables presented in Abramowitz [1]. In these tables the factor $\frac{1}{2 \pi i}$ is included in the definition of the inverse Laplace transform. This is exactly why we included the factor from the beginning. Employing Abramowitz [1] (or employing the computer system Maple [?]) we obtain for $F(t)$ :

$$
\begin{align*}
F(t)= & 2 \frac{a \beta_{k} \sin \left(\beta_{k} t\right)+\left(\omega_{0}^{2}-\beta_{k}^{2}+a^{2}\right) \cos \left(\beta_{k} t\right)}{\left[\left(\beta_{k}+\omega_{0}\right)^{2}+a^{2}\right]\left[\left(\beta_{k}-\omega_{0}\right)^{2}+a^{2}\right]} \\
& +\frac{1}{2} \frac{\left(\omega_{0}-i a\right) e^{-\left(i \omega_{0}+a\right) t}}{\omega_{0}\left[\left(\beta_{k}+\omega_{0}\right)-i a\right]\left[\left(\beta_{k}-\omega_{0}\right)+i a\right]} \\
& +\frac{1}{2} \frac{\left(\omega_{0}+i a\right) e^{\left(i \omega_{0}-a\right) t}}{\omega_{0}\left[\left(\beta_{k}+\omega_{0}\right)+i a\right]\left[\left(\beta_{k}-\omega_{0}\right)-i a\right]} . \tag{6.108}
\end{align*}
$$

For $a=0$ (no added damping) Eq.(6.108) simplifies to the more manageable solution:

$$
\begin{equation*}
F(t)_{a=0}=\frac{\cos \left(\omega_{0} t\right)-\cos \left(\beta_{k} t\right)}{\beta_{k}^{2}-\omega_{0}^{2}} \tag{6.109}
\end{equation*}
$$

The inverse Laplace transform of $G(s)$ can be obtained employing the tables presented in Abramowitz [1]:

$$
\begin{equation*}
G(t, \bar{x})=-\frac{\beta_{k} \bar{x}}{\sqrt{t^{2}-\bar{x}^{2}}} J_{1}\left(\beta_{k} \sqrt{t^{2}-\bar{x}^{2}}\right) H(t-\bar{x})+\delta(t-\bar{x}) \tag{6.110}
\end{equation*}
$$

Writing out the convolution integral (Eq.(6.107)) for $a=0$ results in:

$$
\begin{equation*}
\left.I(\tau, \bar{x}, t)\right|_{a=0}=-\frac{\beta_{k} \bar{x}}{\beta_{k}^{2}-\omega_{0}^{2}} \int_{\bar{x}}^{t}\left[\cos \left(\omega_{0}(t-\tau)\right)-\cos \left(\beta_{k}(t-\tau)\right)\right] \frac{J_{1}\left(\beta_{k} \sqrt{\tau^{2}-\bar{x}^{2}}\right)}{\sqrt{\tau^{2}-\bar{x}^{2}}} d \tau \tag{6.111}
\end{equation*}
$$

For $t<\bar{x}$ the result is zero. The contribution of the delta function has to be omitted because it is non-causal. From Eq.(6.111) it can be seen that for $k \rightarrow \infty$ we have $\left.I(\tau, \bar{x}, t)\right|_{a=0} \rightarrow 0$. Taking into account the different retarded times for $x \pm l$, we obtain for $p_{k}$ :

$$
\begin{equation*}
p_{k}(x, t)=-\omega_{0}\left\{I(\tau, x-l, t) H\left(\tau_{1}\right)-I(\tau, x+l, t) H\left(\tau_{2}\right)\right\} . \tag{6.112}
\end{equation*}
$$

The solution for $p_{k}$ given by Eq.(6.112) is valid for $\bar{x}>l$ and is applicable in both the near field of the vibrating plate and the far field $\bar{x} \gg l$. Unfortunately the solution for $p_{k}$ involves the evaluation of a convolution integral, which we have not been able to evaluate analytically. Numerical approximation of $p_{k}$ does not seem to be a good alternative either, since we have to evaluate an infinite number of integrals, of which the integrands are comprised of combinations of trigonometric functions, Bessel functions and powers. Bessel functions are nice and
short notations describing infinite series, but not easily represented numerically. Clearly the obtained solution is not very practical. It is noted, though, that Eq.(6.112) can be evaluated for $t \rightarrow \infty$ employing Gradshteyn \& Ryzhik [36].

Fortunately it is possible to obtain a far-field approximation of Eq.(6.99), which we will describe next.

When we are interested in the far-field solution the integral can be approximated employing the method of Stationary Phase, see appendix C.4, assuming both $x$ and $t$ large, with $t>x$. Let us write the integral of Eq.(6.99) in the form of appendix C.4:

$$
\begin{equation*}
I(\omega, \bar{x}, t)=\frac{1}{2 \pi i} \int_{-\infty}^{\infty} g(\omega) e^{i t \phi(\omega, \bar{x}, t)} d \omega \tag{6.113}
\end{equation*}
$$

where

$$
\begin{align*}
& \phi(\omega, \bar{x}, t)=\omega-\frac{\bar{x}}{t} \lambda_{0 k}(\omega)  \tag{6.114}\\
& g(\omega)=\frac{-\omega}{\lambda_{0 k}^{2}\left[\omega-\left(\omega_{0}+i a\right)\right]\left[\omega-\left(-\omega_{0}+i a\right)\right]} . \tag{6.115}
\end{align*}
$$

Let us now obtain the value of $\omega$ for which the slope of $\phi^{\prime}(\omega)$ equals zero:

$$
\begin{equation*}
\omega=\omega^{*} \equiv \frac{\beta_{k}}{\sqrt{1-\left(\frac{\bar{x}}{t}\right)^{2}}} \tag{6.116}
\end{equation*}
$$

Note that $\omega=-\omega^{*}$, which is also a root of $\phi^{\prime}(\omega)$, actually does not satisfy, i.e $\phi^{\prime}\left(-\omega^{*}\right) \neq 0$. So, $\omega=\omega^{*}$ is the only, so-called, stationary point of $\phi$. Furthermore we know that $t>\bar{x}$ (where both $t$ and $\bar{x}$ are large). Employing appendix C.4, we obtain the approximation:

$$
\begin{equation*}
I(\omega, \bar{x}, t) \approx \frac{1}{2 \pi i} \hat{I}\left(\omega^{*}, \bar{x}, t\right) \tag{6.117}
\end{equation*}
$$

with:

$$
\begin{equation*}
\hat{I}\left(\omega^{*}, \bar{x}, t\right) \equiv g\left(\omega^{*}\right) \sqrt{\frac{2 \pi}{t\left|\phi^{\prime \prime}\left(\omega^{*}\right)\right|}} e^{i\left[t \phi\left(\omega^{*}\right)+\pi / 4\right]} \tag{6.118}
\end{equation*}
$$

Like $p_{0}$, see Eq.(6.95), the solution for $p_{k}$ exists of two parts, one for $x-l$ and one for $x+l$. The former leads to a signal with retarded time $\tau_{1}=t-(x-l)$, the latter to a signal with retarded time $\tau_{2}=t-(x+l)$. Taking into account the retarded times, we obtain the following approximation for $p_{k}$ :

$$
\begin{equation*}
p_{k}(x, t) \approx-\frac{\omega_{0}}{2 \pi} \operatorname{Re}\left\{-i \hat{I}\left(\omega^{*}, x-l, t\right) H\left(\tau_{1}\right)+i \hat{I}\left(\omega^{*}, x+l, t\right) H\left(\tau_{2}\right)\right\} \tag{6.119}
\end{equation*}
$$

## Complete solution

The complete, approximate, solution for the vibrating wall problem with forcing function given by Eq.(6.75) can be written as:

$$
\begin{equation*}
p(x, z, t) \cong p_{0}(x, t)+\sum_{k=1}^{\infty} \cos (k \pi z) p_{k}(x, t) \tag{6.120}
\end{equation*}
$$

For $z=\frac{1}{2}$ the solution simplifies:

$$
\begin{equation*}
p\left(x, \frac{1}{2}, t\right)=p_{0}(x, t)+\sum_{k=1}^{\infty}(-1)^{k} p_{2 k}(x, t) \tag{6.121}
\end{equation*}
$$

### 6.4.2 Specifi c case B

Let us now consider for $\psi$ the function:

$$
\begin{equation*}
\psi(x, t)=\cos \left(\chi_{n} x\right) \sin \left(\omega_{0} t\right) H(t) \tag{6.122}
\end{equation*}
$$

where

$$
\begin{equation*}
\chi_{n}=\frac{n \pi}{2 l}, \quad n \in \mathbb{N} \tag{6.123}
\end{equation*}
$$

We will only consider cases in which $n$ is odd, which results in a situation where the plate is continuous at $x \pm l$, see also Fig.(6.6). As in the previous section, section 6.4.1, the solution


Figure 6.6: Side view of vibrating wall segment.
is independent of $y$ (so only for $m=0$ we have a non-zero contribution) and can be written in the form of Eq.(6.120):

$$
\begin{equation*}
p^{b}(x, z, t)=p_{0}^{b}(x, t)+\sum_{k=1}^{\infty} \cos (k \pi z) p_{k}^{b}(x, t) \tag{6.124}
\end{equation*}
$$

where we have added superscripts $b$ to distinguish the results from the ones obtained in the preceding section. The solution procedures to obtain $p_{0}^{b}$ and $p_{k}^{b}$ are similar to the ones presented in the previous section. Because of the similarities the description of the solution procedure, especially the one for $p_{0}^{b}$, will be somewhat more brief here.

## Evaluation of $p_{0}^{b}$

For $p_{0}^{b}$ we have to evaluate:

$$
\begin{equation*}
p_{0}^{b}(x, t)=\frac{\chi_{n} \omega_{0} \sin \left(\chi_{n} l\right)}{4 \pi} \int_{-\infty}^{\infty} \frac{e^{-i(x-l) \omega}+e^{-i(x+l) \omega}}{\left(\omega^{2}-\omega_{0}^{2}\right)\left(\omega^{2}-\chi_{n}^{2}\right)} e^{i \omega t} d \omega \tag{6.125}
\end{equation*}
$$

which can be shown to result in:

$$
\begin{equation*}
p_{0}^{b}(x, t)=-\frac{1}{2} \frac{\sin \left(\chi_{n} l\right)}{\chi_{n}^{2}-\omega_{0}^{2}}\left\{g^{b}\left(\tau_{1}\right) H\left(\tau_{1}\right)+g^{b}\left(\tau_{2}\right) H\left(\tau_{2}\right)\right\}, \tag{6.126}
\end{equation*}
$$

where $\sin \left(\chi_{n} l\right)=(-1)^{\frac{n-1}{2}}$ and

$$
\begin{equation*}
g^{b}(\tau)=\omega_{0} \sin \left(\chi_{n} \tau\right)-\chi_{n} \sin \left(\omega_{0} \tau\right) . \tag{6.127}
\end{equation*}
$$

The retarded times $\tau_{1}$ and $\tau_{2}$ are given by Eq.(6.97) and (6.98):

$$
\tau_{1}=t-(x-l), \quad \tau_{2}=t-(x+l)
$$

## Evaluation of $p_{k}^{b}$

For $p_{k}^{b}$ we have to evaluate:

$$
\begin{equation*}
p_{k}^{b}(x, t)=-\frac{\chi_{n} \omega_{0} \sin \left(\chi_{n} l\right)}{2 \pi} \int_{-\infty}^{\infty} \frac{\omega\left(e^{-i(x-l) \lambda_{0 k}}+e^{-i(x+l) \lambda_{0 k}}\right)}{\lambda_{0 k}\left(\omega^{2}-\omega_{0}^{2}\right)\left(\omega^{2}-\beta_{k}^{2}-\chi_{n}^{2}\right)} e^{i \omega t} d \omega \tag{6.128}
\end{equation*}
$$

It has been found beneficial to solve the above integral employing Laplace transformation. Upon substitution of $\omega=s / i$ in Eq.(6.128), the integral we have to evaluate (twice, once for $\bar{x}=x-l$ and once for $\bar{x}=x+l$ ) can be written in the form of Eq.(6.103):

$$
\begin{equation*}
I^{b}(s, \bar{x}, t)=\frac{1}{2 \pi i} \int_{-i \infty}^{i \infty} F^{b}(s) G^{b}(s, \bar{x}) e^{s t} d s \tag{6.129}
\end{equation*}
$$

where this time:

$$
\begin{align*}
& F^{b}(s)=\frac{s}{\left[s^{2}+\omega_{0}^{2}\right]\left[s^{2}+\phi_{k n}^{2}\right]}, \quad \phi_{k n}^{2}=\beta_{k}^{2}+\chi_{n}^{2}  \tag{6.130}\\
& G^{b}(s, \bar{x})=\frac{e^{-\bar{x} \sqrt{s^{2}+\beta_{k}^{2}}}}{\sqrt{s^{2}+\beta_{k}^{2}}} \tag{6.131}
\end{align*}
$$

Note again the minus sign in the exponent of $G^{b}(s)$, see also the preceding section for an explanation. To solve Eq.(6.129) we can apply the convolution theorem, from which it is obtained:

$$
\begin{equation*}
I^{b}(\tau, \bar{x}, t)=\int_{0}^{t} F^{b}(t-\tau) G^{b}(\tau, \bar{x}) d \tau \tag{6.132}
\end{equation*}
$$

Working out the inverse Laplace transform of $F^{b}(s)$ to $F^{b}(t)$ we obtain:

$$
\begin{equation*}
F^{b}(t)=\frac{\cos \left(\omega_{0} t\right)-\cos \left(\phi_{k n} t\right)}{\phi_{k n}^{2}-\omega_{0}^{2}} \tag{6.133}
\end{equation*}
$$

The inverse Laplace transform of $G^{b}(s, \bar{x})$ to $G^{b}(t)$ can be obtained from Abramowitz [1]:

$$
\begin{equation*}
G^{b}(t, \bar{x})=J_{0}\left(\beta_{k} \sqrt{t^{2}-\bar{x}^{2}}\right) H(t-\bar{x}) \tag{6.134}
\end{equation*}
$$

From the above results we obtain for Eq.(6.132):

$$
\begin{equation*}
I^{b}(\tau, \bar{x}, t)=\frac{1}{\phi_{k n}^{2}-\omega_{0}^{2}} \int_{\bar{x}}^{t}\left[\cos \left(\omega_{0}(t-\tau)\right)-\cos \left(\phi_{k n}(t-\tau)\right)\right] J_{0}\left(\beta_{k} \sqrt{\tau^{2}-\bar{x}^{2}}\right) d \tau \tag{6.135}
\end{equation*}
$$

For $t<\bar{x}$ the result is zero. Taking into account the different retarded times for $x \pm l$, i.e. $\tau_{1}=t-(x-l)$ and $\tau_{2}=t-(x+l)$ we obtain the for $p_{k}^{b}$ :

$$
\begin{equation*}
p_{k}^{b}(x, t)=-\frac{\omega_{0} \chi_{n} \sin \left(\chi_{n} l\right)}{\phi_{k n}^{2}-\omega_{0}^{2}}\left\{I^{b}(\tau, x-l, t) H\left(\tau_{1}\right)+I^{b}(\tau, x+l, t) H\left(\tau_{2}\right)\right\} \tag{6.136}
\end{equation*}
$$

Unlike the solution obtained in the preceding section, Eq.(6.135) can be manipulted, such that it will provide us an expression from which we can derive the exact solution for $t \rightarrow \infty$. Let us write Eq.(6.135) in the short form:

$$
\begin{equation*}
I^{b}(\tau, \bar{x}, t)=C_{k}\left\{I_{1}(\tau, \bar{x}, t)-I_{2}(\tau, \bar{x}, t)\right\}, \tag{6.137}
\end{equation*}
$$

where

$$
\begin{align*}
& C_{k}=\frac{1}{\phi_{k n}^{2}-\omega_{0}^{2}}  \tag{6.138}\\
& I_{i}(\tau, \bar{x}, t)=\int_{\bar{x}}^{t} \cos \left(\varphi_{i}(t-\tau)\right) J_{0}\left(\beta_{k} \sqrt{\tau^{2}-\bar{x}^{2}}\right) d \tau, \quad i=1,2 \tag{6.139}
\end{align*}
$$

where

$$
\begin{equation*}
\varphi_{1}=\omega_{0}, \quad \varphi_{2}=\phi_{k n} \tag{6.140}
\end{equation*}
$$

Employing trigoniometric relations, we can write:

$$
\begin{equation*}
I_{i}(\tau, \bar{x}, t)=\cos \left(\varphi_{i} t\right) I_{i}^{a}(\tau, \bar{x}, t)+\sin \left(\varphi_{i} t\right) I_{i}^{b}(\tau, \bar{x}, t) \tag{6.141}
\end{equation*}
$$

where:

$$
\begin{align*}
I_{i}^{a}(\tau, \bar{x}, t) & =\int_{\bar{x}}^{t} \cos \left(\varphi_{i} \tau\right) J_{0}\left(\beta_{k} \sqrt{\tau^{2}-\bar{x}^{2}}\right) d \tau  \tag{6.142}\\
I_{i}^{b}(\tau, \bar{x}, t) & =\int_{\bar{x}}^{t} \sin \left(\varphi_{i} \tau\right) J_{0}\left(\beta_{k} \sqrt{\tau^{2}-\bar{x}^{2}}\right) d \tau \tag{6.143}
\end{align*}
$$

Next we split the integrals into an integral over $\tau=[\bar{x}, \infty)$ minus an integral over $\tau=[t, \infty)$ :

$$
\begin{align*}
& I_{i}^{a}(\tau, \bar{x}, t)=\int_{\bar{x}}^{\infty} \cos \left(\varphi_{i} \tau\right) J_{0}\left(\beta_{k} \sqrt{\tau^{2}-\bar{x}^{2}}\right) d \tau-\int_{t}^{\infty} \cos \left(\varphi_{i} \tau\right) J_{0}\left(\beta_{k} \sqrt{\tau^{2}-\bar{x}^{2}}\right) d \tau,(6  \tag{6.144}\\
& I_{i}^{b}(\tau, \bar{x}, t)=\int_{\bar{x}}^{\infty} \sin \left(\varphi_{i} \tau\right) J_{0}\left(\beta_{k} \sqrt{\tau^{2}-\bar{x}^{2}}\right) d \tau-\int_{t}^{\infty} \sin \left(\varphi_{i} \tau\right) J_{0}\left(\beta_{k} \sqrt{\tau^{2}-\bar{x}^{2}}\right) d \tau \tag{6.145}
\end{align*}
$$

From Gradshteyn and Ryzhik [36] we have:

$$
\begin{align*}
& \int_{\bar{x}}^{\infty} \cos \left(\varphi_{i} \tau\right) J_{0}\left(\beta_{k} \sqrt{\tau^{2}-\bar{x}^{2}}\right) d \tau= \begin{cases}\frac{e^{-\bar{x} \sqrt{\beta_{k}^{2}-\varphi_{i}^{2}}}}{\sqrt{\beta_{k}^{2}-\varphi_{i}^{2}}}, & 0<\varphi_{i}<\beta_{k} \\
\frac{\sin \left(\bar{x} \sqrt{\varphi_{i}^{2}-\beta_{k}^{2}}\right)}{\sqrt{\varphi_{i}^{2}-\beta_{k}^{2}}}, & 0<\beta_{k}<\varphi_{i}\end{cases}  \tag{6.146}\\
& \int_{\bar{x}}^{\infty} \sin (\varphi \tau) J_{0}\left(\beta_{k} \sqrt{\tau^{2}-\bar{x}^{2}}\right) d \tau= \begin{cases}0, & 0<\varphi_{i}<\beta_{k} \\
\frac{\cos \left(\bar{x} \sqrt{\varphi_{i}^{2}-\beta_{k}^{2}}\right)}{\sqrt{\varphi_{i}^{2}-\beta_{k}^{2}}}, & 0<\beta_{k}<\varphi_{i}\end{cases} \tag{6.147}
\end{align*}
$$

From Eq.(6.146) and Eq.(6.147) we observe that for $t \rightarrow \infty$ the solution $p_{k}^{b}$ becomes harmonic.

Assuming $\omega_{0}<\pi$, so $\omega_{0}<\beta_{k} \forall k$, we obtain for $t \rightarrow \infty$ :

$$
\begin{align*}
\lim _{t \rightarrow \infty} p_{k}^{b}(x, t) & =-\frac{\omega_{0} \sin \left(\chi_{n} l\right)}{2 \pi\left(\phi_{k n}^{2}-\omega_{0}^{2}\right)}\{ \\
& \frac{\cos \left(\omega_{0} t\right)}{\sqrt{\beta_{k}^{2}-\omega_{0}^{2}}}\left[e^{-(x-l) \sqrt{\beta_{k}^{2}-\omega_{0}^{2}}} H\left(\tau_{1}\right)+e^{-(x+l) \sqrt{\beta_{k}^{2}-\omega_{0}^{2}}} H\left(\tau_{2}\right)\right] \\
& -\frac{\cos \left(\phi_{k n} t\right)}{\chi_{n}}\left[\sin \left(\chi_{n}(x-l)\right) H\left(\tau_{1}\right)+\sin \left(\chi_{n}(x+l)\right) H\left(\tau_{2}\right)\right] \\
& -\frac{\sin \left(\phi_{k n} t\right)}{\chi_{n}}\left[\cos \left(\chi_{n}(x-l)\right) H\left(\tau_{1}\right)+\cos \left(\chi_{n}(x+l)\right) H\left(\tau_{2}\right)\right\} . \tag{6.148}
\end{align*}
$$

At a location resonably far away from the vibrating wall, $x \gg l$, we observe for the pressure for $t \rightarrow \infty$, assuming $\tau_{1}>0$ and $\tau_{2}>0$ :

$$
\begin{align*}
\lim _{t \rightarrow \infty} p_{k}^{b}(x, t)= & \frac{\omega_{0} \sin \left(\chi_{n} l\right)}{\pi \chi_{n}\left(\phi_{k n}^{2}-\omega_{0}^{2}\right)} \sin \left(\chi_{n} l\right) \cos \left(\chi_{n} l\right) \\
& \left\{\cos \left(\phi_{k n} t\right) \sin \left(\chi_{n} x\right)+\sin \left(\phi_{k n} t\right) \cos \left(\chi_{n} x\right)\right\}=0 \tag{6.149}
\end{align*}
$$

The result of Eq.(6.149) is zero because $\cos \left(\chi_{n} l\right)=0$ for $n$ an odd integer. It appears from Eq.(6.149) that eventually for $t \rightarrow \infty$ the contribution of $p_{k}^{b}$ to the solution for the pressure perturbation disappears (for $x \gg l$ ), Hso that only the contribution $p_{0}^{b}$ remains.

It is furthermore noted that the assumption $\omega_{0}<\pi$ is not unreasonable. If, for example, we assume the height of the duct to be $h=0.25 m$ (e.g. ventilation channel) this assumption holds for frequencies of the vibrating wall for up to approximately 4 kHz . Because we are also interested in the numerical solution, we will have to restrict ourselves to moderate frequencies. This because for high frequencies the mesh has to be fine enough for spatial resolution.

### 6.5 Radiation from vibrating wall segment in duct with uniform mean fbw

Let us now consider the same problem as considered in section 6.4, only this time we
assume that there is a uniform mean flow in $x$-direction in the duct, i.e. $M \neq 0$. The meanflow velocities, expressed in terms of Mach numbers, are now given by $\mathbf{u}_{0}=(M, 0,0)^{T}$. Under this assumption the dimensionless acoustic density perturbation $\rho$, velocity field $\mathbf{u}=$ $(u, v, w)^{T}$ and pressure perturbation $p$ satisfy the following linearized continuity equation, linearized momentum equation and linearized energy equation (isentropic flow):

$$
\begin{align*}
& \frac{\partial \rho}{\partial t}+M \frac{\partial \rho}{\partial x}+\frac{\partial u_{j}}{\partial x_{j}}=0  \tag{6.150}\\
& \frac{\partial u_{i}}{\partial t}+M \frac{\partial u_{i}}{\partial x}+\frac{\partial p}{\partial x_{i}}=0 \tag{6.151}
\end{align*}
$$

and

$$
\begin{equation*}
p=\rho \tag{6.152}
\end{equation*}
$$

respectively.
From Eqs.(6.150), (6.151) and (6.152) the convected wave equation can be derived for the pressure:

$$
\begin{equation*}
\frac{D^{2} p}{D t^{2}}-\nabla^{2} p=0, \quad \frac{D}{D t}=\frac{\partial}{\partial t}+M \frac{\partial}{\partial x} \tag{6.153}
\end{equation*}
$$

From Eq.(6.10), we have:

$$
\begin{equation*}
w=\frac{\partial z_{s}}{\partial t}+M \frac{\partial z_{s}}{\partial x} \tag{6.154}
\end{equation*}
$$

where $w=w^{\prime} / \epsilon$ and $z_{s}=z_{s}^{\prime} / \epsilon$. Taking the inner product of the momentum equation in $z-$ direction with the unit normal $\mathbf{n}_{0}=-\mathbf{e}_{z}$, results in the following linearized conditions for the pressure:

$$
\begin{equation*}
\left.\frac{\partial p}{\partial z}\right|_{z=0}=\frac{\partial u_{n}}{\partial t}+M \frac{\partial u_{n}}{\partial x} \tag{6.155}
\end{equation*}
$$

where $u_{n}=-w$. The hard-wall boundary condition applies to all the other walls of the duct, i.e. $y=0, y=b$ and $z=1$.

Fourier transforming in both $t$ and $x$, recasts Eq.(6.153) into:

$$
\begin{equation*}
\frac{\partial^{2} \hat{p}}{\partial y^{2}}+\frac{\partial^{2} \hat{p}}{\partial z^{2}}+\left[(\omega-M \xi)^{2}-\xi^{2}\right] \hat{p}=0, \quad \hat{p}=\hat{p}(\xi, y, z, \omega) \tag{6.156}
\end{equation*}
$$

which shows much analogy with the Helmholtz equation of Eq.(6.38). Fourier transformation of Eq.(6.155) results in:

$$
\begin{equation*}
\left.\frac{\partial \hat{p}}{\partial z}\right|_{z=0}=i(\omega-M \xi) \hat{u}_{n} \tag{6.157}
\end{equation*}
$$

It is noted that the coordinate transformation of chapter 5 could have been used to transform the convected wave equation of Eq.(6.153) into the "ordinary" wave equation of Eq.(6.16). This particular transformation unfortunately complicates the boundary condition for $z=0$. Alternatively a Lorentz type transformation, where $\xi=\left(\xi^{*}-M \omega^{*}\right) / \beta, \omega=\beta \omega^{*}$ and $\beta=$ $\sqrt{1-M^{2}}$, can be employed to transform Eq.(6.156) into the "ordinary" Helmholtz equation of Eq.(6.21). However, this transformation seems to complicate the Fourier transformations which are necessary to obtain the solution.

Eq.(6.156) can be solved together with the boundary conditions in a similar way as presented in section 6.4 for the case without mean flow. Because of the great similarity with the
solution procedure of section 6.4 , the description will be shorter here, i.e. only some of the intermediate results will be presented.

It can easily be shown that the solution of Eq.(6.156), which adheres to the boundary condition given by Eq.(6.157) and the hard-wall conditions, is given by:

$$
\begin{equation*}
\hat{p}(\xi, y, z, \omega)=i \sum_{m=0}^{\infty} \cos \left(\alpha_{m} y\right)(\omega-M \xi) \hat{u}_{n_{m}}(\xi, \omega) \frac{\cos \left((z-1) \gamma_{m}^{*}\right)}{\gamma_{m}^{*} \sin \left(\gamma_{m}^{*}\right)} \tag{6.158}
\end{equation*}
$$

where

$$
\begin{equation*}
\gamma_{m}^{*}=\sqrt{(\omega-M \xi)^{2}-\xi^{2}-\alpha_{m}^{2}}, \quad \alpha_{m}=\frac{m \pi}{b} \tag{6.159}
\end{equation*}
$$

From Eq.(6.158) we obtain for the pressure in the frequency domain:
$\tilde{p}(x, y, z, \omega)=\frac{i \omega}{2 \pi} \sum_{m=0}^{\infty} \cos \left(\alpha_{m} y\right) \int_{-l}^{l} \int_{-\infty}^{\infty}(\omega-M \xi) \frac{\cos \left((z-1) \gamma_{m}^{*}\right)}{\gamma_{m}^{*} \sin \left(\gamma_{m}^{*}\right)} e^{-i\left(x-x^{\prime}\right) \xi} d \xi \tilde{\psi}_{m}\left(x^{\prime}, \omega\right) d x^{\prime}$.

If we denote the integrand for the inverse Fourier transform from $\xi$ to $x$ in the above equation by $(\omega-M \xi) I(\xi)$, it is clear we have to evaluate two inverse Fourier transforms here, one for $\omega I(\xi)$ and one for $M \xi I(\xi)$. The poles of both integrands are given by $\gamma_{m}^{*}=0$ :

$$
\begin{equation*}
\xi=\Upsilon_{m}^{ \pm}(M, \omega), \tag{6.161}
\end{equation*}
$$

where

$$
\begin{align*}
\Upsilon_{m}^{ \pm}(M, \omega) & \equiv \frac{ \pm \Omega_{m}(M, \omega)-M \omega}{1-M^{2}}  \tag{6.162}\\
\Omega_{m}^{2}(M, \omega) & \equiv \omega^{2}-\left(1-M^{2}\right) \alpha_{m}^{2} \tag{6.163}
\end{align*}
$$

and by $\gamma_{m}^{*}=k \pi$ :

$$
\begin{equation*}
\xi=\Lambda_{m k}^{ \pm}(M, \omega) \tag{6.164}
\end{equation*}
$$

where

$$
\begin{align*}
& \Lambda_{m k}^{ \pm}(M, \omega) \equiv \frac{ \pm \lambda_{m k}(M, \omega)-M \omega}{1-M^{2}}  \tag{6.165}\\
& \lambda_{m k}(M, \omega) \equiv \omega^{2}-\left(1-M^{2}\right)\left(\alpha_{m}^{2}+\beta_{k}^{2}\right) \tag{6.166}
\end{align*}
$$

The first integral, i.e. for $\omega I(\xi)$, is very similar to the one evaluated in section 6.4 , with the poles shifted in the complex $\zeta$-plane, where $\operatorname{Re}(\zeta)=\xi$. The poles of the integrand in section 6.4 were symmetric in the sense that the poles enclosed by the contour $C^{-}$are obtained from the poles enclosed by $C^{+}$by simply changing the signs. In the case $M \neq 0$ the poles are no longer symmetric in this sense.

We will only present the solution for $x-x^{\prime}>0$. For the solution procedure for evaluating the inverse Fourier transforms the reader is referred to the procedure presented in section 6.4.

The solution for $x-x^{\prime}>0$ for the pressure in the frequency domain can be shown to be given by:

$$
\begin{align*}
& \tilde{p}(x, y, z, \omega)= \sum_{m=0}^{\infty} \\
& \cos \left(\alpha_{m} y\right)\left\{\frac{M \Upsilon_{m}(M, \omega)-\omega}{2 \Omega_{m}(M, \omega)} \int_{-l}^{l} \tilde{\psi}_{m}\left(x^{\prime}, \omega\right) e^{-i\left(x-x^{\prime}\right) \Upsilon_{m}(M, \omega)} d x^{\prime}+\right.  \tag{6.167}\\
&\left.\sum_{k=1}^{\infty} \cos \left(\beta_{k} z\right) \frac{M \Lambda_{m k}(M, \omega)-\omega}{\lambda_{m k}(M, \omega)} \int_{-l}^{l} \tilde{\psi}_{m}\left(x^{\prime}, \omega\right) e^{-i\left(x-x^{\prime}\right) \Lambda_{m k}(M, \omega)} d x^{\prime}\right\}
\end{align*}
$$

where again:

$$
\beta_{k}=k \pi
$$

The solution shows great analogy with the solution presented in Eq.(6.67). As expected, we observe that for $M=0$ we have:

$$
\begin{align*}
& \Upsilon_{m}(0, \omega)=\Omega_{m}  \tag{6.168}\\
& \Lambda_{m k}(0, \omega)=\lambda_{m k} \tag{6.169}
\end{align*}
$$

where $\Omega_{m}$ and $\lambda_{m k}$ have been defined in Eq.(6.48) and Eq.(6.49) of section 6.4, respectively. For $M=0$ Eq.(6.167) simplifies to Eq.(6.67).

### 6.5.1 Specifi c case A with uniform mean flow

Now consider how the uniform mean flow velocity in $x$-direction affects the solution for the pressure for the specific case presented in section 6.4.1. In this section we will only consider $p_{0}$.

The normal velocity vibration function $\psi$ is given by Eq.(6.75). For this specific case we only have a non-trivial solution for $m=0$, see also section 6.4.1. It can be shown that:

$$
\begin{equation*}
\tilde{p}_{0}(x, \omega)=\frac{M \Upsilon_{0}(M, \omega)-\omega}{2 \Omega_{0}(M, \omega)} \int_{-l}^{l} \tilde{\psi}_{0}(\omega) e^{-i\left(x-x^{\prime}\right) \Upsilon_{0}(M, \omega)} d x^{\prime} \tag{6.170}
\end{equation*}
$$

where

$$
\begin{array}{ll}
\Omega_{0}(M, \omega)=\omega, & \Upsilon_{0}(M, \omega)=\frac{(1-M) \omega}{1-M^{2}}=\frac{\omega}{1+M} \\
\tilde{\psi}_{0}(\omega)=Q(\omega), & \tilde{\psi}_{m}(\omega)=0, \quad \text { for } m>0 \tag{6.172}
\end{array}
$$

and where $Q(\omega)$ is given by Eq.(6.76). To obtain $p_{0}$ we have to evaluate:

$$
\begin{equation*}
p_{0}(x, t)=-\frac{\omega_{0}}{4 \pi i} \int_{-\infty}^{\infty} \frac{e^{-i(1+M)^{-1}(x-l) \omega}-e^{-i(1+M)^{-1}(x+l) \omega}}{\omega\left[i\left(\omega-\omega_{0}\right)+a\right]\left[i\left(\omega+\omega_{0}\right)+a\right]} e^{i \omega t} d \omega \tag{6.173}
\end{equation*}
$$

We can now simply use the result from evaluating Eq.(6.83), where, this time the retarded times are given by:

$$
\begin{align*}
\tau_{1} & =t-\frac{x-l}{1+M}  \tag{6.174}\\
\tau_{2} & =t-\frac{x+l}{1+M} \tag{6.175}
\end{align*}
$$

So as a result we obtain for $p_{0}$ :

$$
\begin{equation*}
p_{0}(x, t)=\frac{1}{2} \frac{\omega_{0}}{\omega_{0}^{2}+a^{2}}\left\{g\left(\tau_{1} ; M\right) H\left(\tau_{1}\right)-g\left(\tau_{2} ; M\right) H\left(\tau_{2}\right)\right\} \tag{6.176}
\end{equation*}
$$

where $g(\tau)$ is given by (see also Eq.(6.96)):

$$
\begin{equation*}
g(\tau ; M)=\left[\cos \left(\omega_{0} \tau\right)+\frac{a}{\omega_{0}} \sin \left(\omega_{0} \tau\right)\right] e^{-a \tau}-1 \tag{6.177}
\end{equation*}
$$

It can be shown that the solution for $x<-l$ is given by Eq.(6.176) where the following retarded times are to be taken instead of $\tau_{1}$ and $\tau_{2}$ :

$$
\begin{align*}
& \tau_{1}^{-}=t+\frac{x+l}{1-M}=t-\frac{|x|-l}{1-M}  \tag{6.178}\\
& \tau_{2}^{-}=t+\frac{x-l}{1-M}=t-\frac{|x|+l}{1-M} \tag{6.179}
\end{align*}
$$

respectively. Note that the solution for $x<-l$ can also obtained from the solution for $x>l$ by simply replacing $x$ by $-x$ and $M$ by $-M$.

The pressure wave represented by Eq.(6.176) is a propagating mode or wave. That is, it is only truly propagating when $a=0$. The obtained solution has been expressed in terms of the "laboratory coordinate system" $(x, t)$ which is fixed at the stationary duct.

In general, a wave propagating in positive $x$-direction, expressed in terms of the laboratory coordinate system $(x, t)$, can be written proportional to:

$$
\begin{equation*}
e^{i(k x-\omega t)} \tag{6.180}
\end{equation*}
$$

where $k=\frac{2 \pi}{\lambda_{x}}$ and $\lambda_{x}$ is the wavelength of the in $x$-direction propagating wave. The solution for $p_{0}$, presented in Eq.(6.176), represents a propagating wave (for $a=0$ ) which is proportional to:

$$
\begin{equation*}
e^{i \omega_{0}\left(\frac{x}{1+M}-t\right)}=e^{i(k x-\omega t)} . \tag{6.181}
\end{equation*}
$$

So, $k=\frac{\omega_{0}}{1+M}$ and $\omega=\omega_{0}$. A propagating pressure wave, expressed in terms of a coordinate system moving with the mean flow, is proportional to ([34], [46], [59]):

$$
\begin{equation*}
e^{i\left(k x^{\prime}-\omega^{\prime} t\right)} \tag{6.182}
\end{equation*}
$$

where

$$
\begin{equation*}
x=x^{\prime}+M t . \tag{6.183}
\end{equation*}
$$

To an observer at rest in the laboratory system $(x, t)$, the pressure will be proportional to

$$
\begin{equation*}
e^{i\left(k x-\left(\omega^{\prime}+k M\right) t\right)}=e^{i(k x-\omega t)} \tag{6.184}
\end{equation*}
$$

from which we obtain the famous Doppler formula ([34], [46], [59]):

$$
\begin{equation*}
\omega_{0}=\omega^{\prime}(1+M), \tag{6.185}
\end{equation*}
$$

which relates the frequency $\frac{\omega^{\prime}}{2 \pi}$ in the system moving with the flow to the frequency $\frac{\omega}{2 \pi}$ in the laboratory system.

The solution for $p_{0}$ for $a=0$ can now be expressed in terms of the coordinate system $\left(x^{\prime}, t^{\prime}\right)$, and is given by:

$$
\begin{align*}
\left.p_{0}\left(x^{\prime}, t^{\prime}\right)\right|_{a=0}=\frac{1+M}{2 \omega^{\prime}}\{ & {\left[\cos \left(t^{\prime}-x^{\prime}-l\right)-1\right] H\left(\frac{t^{\prime}-x^{\prime}-l}{1+M}\right) } \\
& \left.-\left[\cos \left(t^{\prime}-x^{\prime}+l\right)-1\right] H\left(\frac{t^{\prime}-x^{\prime}+l}{1+M}\right)\right\} \tag{6.186}
\end{align*}
$$

### 6.5.2 Specifi c case $B$ with uniform mean flow

Consider the case where $\psi$ is given by Eq.(6.122). For this specific case we only have a non-trivial solution for $m=0$, see also section 6.4.2. It can be shown that Eq.(6.170) is again applicable here, only this time $\tilde{\psi}_{0}$ is also a function of $x$ :

$$
\begin{equation*}
\tilde{\psi}_{0}(x, \omega)=-\frac{\omega_{0}}{2 \pi\left(\omega^{2}-\omega_{0}^{2}\right)} \cos \left(\chi_{n} x\right) \tag{6.187}
\end{equation*}
$$

$\tilde{p}_{0}(x, \omega)$ can be evaluated applying Eq.(6.170), where $\tilde{\psi}_{0}$ is now given by Eq.(6.187). Eventually we obtain for $p_{0}$ :

$$
\begin{equation*}
p_{0}^{b}(x, z, t)=-\frac{1}{2} \frac{\sin \left(\chi_{n} l\right)}{\left((1+M) \chi_{n}\right)^{2}-\omega_{0}^{2}}\left\{g^{b}\left(\tau_{1} ; M\right) H\left(\tau_{1}\right)+g^{b}\left(\tau_{2} ; M\right) H\left(\tau_{2}\right)\right\} \tag{6.188}
\end{equation*}
$$

where

$$
\begin{equation*}
g^{b}(\tau ; M)=\omega_{0} \sin \left((1+M) \chi_{n} \tau\right)-(1+M) \chi_{n} \sin \left(\omega_{0} \tau\right) \tag{6.189}
\end{equation*}
$$

and where the retarded times $\tau_{1}$ and $\tau_{2}$ are given by Eqs.(6.174) and (6.175). For $M=0$ we retrieve the solution of section 6.4.2.

# NUMERICAL RESULTS VIBRATING WALL 

### 7.1 Introduction

In this chapter numerical results for the vibrating wall problem inside an infinite rectangular duct are presented. The numerical results have been obtained with the computational method described in chapter 3. For the verification of the computational method the numerical results are compared with the analytical solutions described in chapter 6 . In this chapter we use the following short-hand notation to describe the analytical solutions of sections 6.4.1 and 6.4.2, which are independent of $y$ :

$$
\begin{equation*}
p(x, z, t)=p_{0}(x, t)+p_{d}(x, z, t), \tag{7.1}
\end{equation*}
$$

where

$$
\begin{equation*}
p_{d}(x, z, t)=\sum_{k=1}^{\infty} \cos (k \pi z) p_{k}(x, t) . \tag{7.2}
\end{equation*}
$$

When the mean flow is absent, the problem, described in chapter 6.2, is symmetrical with respect to the plane $x=0$, which has also been verified numerically. For this verification case ( $M=0$ ) the dimensionless linearized Euler equations are solved on a rectangular domain, given by $x \in\left[0, x_{\max }\right], y \in[0,1]$ and $z \in[0,1]$, where all lengths are nondimensional. For $x_{\max }$ a location is chosen sufficiently far away from the vibrating wall. For $M \neq 0$, the problem is no longer symmetrical and the linearized Euler equations are solved on the domain $x \in\left[-x_{\max }, x_{\max }\right], y=[0,1]$ and $z=[0,1]$.

The simulations have been carried out on tetrahedral meshes. In order to obtain the computational grid, the physical domain is first partitioned into equally sized cubes which in turn are all divided into twelve identical tetrahedrons. In this way a reasonably regular tetrahedral mesh is obtained. For all the problems which have been considered a grid-convergence study has been conducted.

During the computations the results are evaluated in all nodes. Each node of the grid is common to at least one, but usually much more than one element.For each element the solution in the node can be evaluated. The node value used for the postprocessing is the average of the node values of the elements sharing the node. Additionally to the node-values at certain pre-determined times, a time history of the perturbation variables is recorded at certain locations throughout the rectangular duct. We will refer to these locations as microphone locations. In each computation between 10 and 15 desired microphone locations are specified
before starting the actual computation. For each of these locations the nearest centroid of an element of the mesh is identified. These specific centroids become the actual microphone locations for which the time history files are created. A drawback of this approach is that the actual microphone location may be at a certain distance from the desired location. However, we do not have to average contributions like we do for the node values.

### 7.2 Results case A

In this section numerical results for the problem described in section 6.4.1 are presented. In the problem of Case A, there are a few parameters which may be varied, i.e. the dimensionless length of the vibrating wall $2 l$, the dimensionless frequency of the vibration $\omega_{0}$ and $a$, which controls the damping of the wall vibration, see also Eq.(6.75). Table 7.1 shows the values of the parameters considered. It is noted that not all combinations of the parameters have been considered.

| parameters Case A |  |  |
| :---: | :---: | :---: |
| $\omega_{0}$ | $l$ | $a$ |
| $\frac{1}{4}$ | 0.6 | 0 |
| $\frac{1}{2}$ | 1 | 0.05 |
| 1 | $\pi$ | 0.1 |
| $\frac{4}{3}$ | 5 |  |
| 2 |  |  |

TABLE 7.1: Values of parameters $\omega_{0}, l$ and a considered for case $A$, where $\omega_{0}$ is the wall vibration frequency, $l$ is (half) the dimensionless length of the vibrating wall and $a$ is controls the damping of the wall vibration.

The results which will be shown in this section have all been obtained, unless stated otherwise, for a computational domain with $x_{\max }=40$, i.e the computational domain is given by $x \in[0,40], y \in[0,1]$ and $z \in[0,1]$. In all cases, except one, the computational domain has been partitioned into tetrahedral meshes consisting of 60000 elements. As mentioned in the introduction of this chapter, the domain is first divided into equally sized cubes, subsequently each cube is divided into twelve identical tetrahedrons. The cubical background mesh dimensions are given by $N_{x}=200, N_{y}=5$ and $N_{z}=5$.

For the vibration frequency $\omega_{0}$, which is a dimensionless number, only moderate values have been considered. Already for a frequency of $\omega_{0}=2$, the number of elements had to be increased (to a number $>60000$ ) because of resolution, increasing the computation time and required storage. Nevertheless, when assuming, for example, a height of the duct of 25 cm (e.g. ventilation channel), the frequencies given in table 7.1 correspond, for air with speed of sound $c_{0}=340 \mathrm{~ms}^{-1}$, to frequencies ranging from (approximately) 340 Hz to 2720 Hz , which is quite reasonable for aeroacoustics.

At the end of section 6.4 it was discussed that it is not easily shown from the solution presented in Eq.(6.69), that the solution near the bottom wall behaves according to the boundary



FIGURE 7.1: Comparison of the velocity component in $z$-direction, $w$, just above the vibrating wall and the imposed normal velocity $\left(u_{n}=-w\right)$. Case $A: \omega_{0}=1$, $a=0$. Microphone location: $x=0.275, y=0.525$ and $z=0.025$.
condition. (Instead it was shown from Eq.(6.42).) Fig.(7.1) shows the velocity in z-direction, $w$, just above the vibrating wall and the imposed normal velocity ( $u_{n}=-w$, normal defined in $-z$-direction) for $\omega_{0}=1, l=1$ and $a=0$. Although the numerical result is obtained a small distance above the wall, where the actual boundary condition is imposed, Fig.(7.1) convincingly shows that the solution for the velocity component $w$ behaves according to the imposed boundary condition.

In Figs.(7.2), (7.3) and (7.4) numerical results and the analytical solution $p_{0}$ are shown for $\omega_{0}=\frac{4}{3}, l=5$ and $a=0.05$. Fig.(7.2) shows results obtained very close to the vibrating wall. In the figure, two numerical results are presented, obtained at two $z$-locations, i.e. $z=0.075$ and $z=0.475$, respectively. The difference between the analytical solution $p_{0}$ and the numerical result for $p$ at $z=0.475$ is observed to be relatively small. At $z=0.075$ the difference is observed to be significant. This can be explained by the following: for $z=\frac{1}{2}$ the analytical solution is given by Eq.(6.121):

$$
p\left(x, \frac{1}{2}, t\right)=p_{0}(x, t)+\sum_{k=1}^{\infty}(-1)^{k} p_{2 k}(x, t)
$$

where $p_{k}$ is given by Eq.(6.112). Employing Eq.(6.111) it can be seen that for $k \rightarrow \infty$ we have $\left|p_{k}\right| \rightarrow 0$. Based on Eq.(6.111), it is furthermore expected that $\left|p_{1}\right|>\left|p_{2}\right|>\ldots>$ $\left|p_{\infty}\right|$. From the results shown in Fig.(7.2) it appears that the contribution of $p_{0}(x, t)$ and $\cos (\pi z) p_{1}(x, t)$ to the solution are most significant.

When the microphone is located near the vibrating wall and not near $z=\frac{1}{2}$ the contribution of $p_{1}(x, z, t)$ to the solution can not be neglected.

Fig.(7.3) shows results for the same problem further downstream of the vibrating wall, at microphone locations in the plane $x=10.075$. Clearly the numerical results, at $z=0.375$


Figure 7.2: Results for Case $A$ at $x=5.025$, for $\omega_{0}=\frac{4}{3}, l=5$ and $a=0.05$.


Figure 7.3: Results for Case $A$ at $x=10.075$, for $\omega_{0}=\frac{4}{3}, l=5$ and $a=0.05$.
and $z=0.825$, show much less deviation from the analytical solution $p_{0}$ than the results presented in Fig.(7.2). In Fig.(7.4) results are shown in the plane $x=20.075$, which is even further away from the vibrating wall. The difference between the numerical result at $z=0.425$ and $p_{0}$ is now no longer visible. Even at $z=0.825$ the numerical result is observed to deviate very little from $p_{0}$. From these results, as well as other results not shown,


Figure 7.4: Results for Case $A$ at $x=20.075$, for $\omega_{0}=\frac{4}{3}, l=5$ and $a=0.05$.
it is concluded that when moving further and further downstream from the vibrating wall, the result for the pressure $p$ gets better and better approximated by the analytical solution $p_{0}$. In Figs.(7.3) and (7.4) the influence of the retarded times $\tau_{1}$ and $\tau_{2}$, Eqs.(6.97) and (6.98), are clearly visible. In Fig.(7.4) nothing happens for $t \in[0, x-l]=[0,15.075]$. At $t=x-l$, the perturbation, created by the wall vibration, has exactly covered the distance from the near end of the vibration wall to the microphone location, travelling at the dimensionless speed of sound $(=1)$. For $t \in[x-l, x+l]=[15.075,25.075]$, so for a period of time $2 l$, only the contribution from the near end of the vibrating wall is "measured" by the microphone. At $t=x+l$, also the contribution from the far end of the vibrating wall has reached the microphone. For $t>x+l$ the contributions from both the near and far end of the vibrating wall are present. From the results shown one might wonder whether the solution for a pressure perturbation created by two point sources at $x= \pm l$ results in a similar result as obtained for the vibrating wall, described above. From the analytical solution which has been obtained for this particular problem (and which will not be shown) it is clear that the answer is no. For the solution $p_{0}$ for the vibrating wall we do distinguish clearly a contribution from both the near and far end of the vibrating wall, however, the entire wall contributes to the solution.

From the analytic solutions for $p_{0}$ presented in section 6.4.1 and the results shown above, it is observed that the solution consists of two major contributions: one originating from the nearest end of the vibrating wall, and one originating from the far end of the vibrating wall. The frequency of the function $g$, which is present in the expression for $p_{0}$ and given by Eq.(6.96), obviously depends on the vibration frequency $\omega_{0}$. The frequency of the solution $p_{0}$, however, also depends on the (dimensionless) length $2 l$ of the vibrating wall. At a certain microphone location $x$ the signal from the far end of the vibrating wall follows the signal from the near end of the vibrating wall $2 l$ dimensionless time-units later. Therefore, depending on the length $2 l$, the two signals can amplify each other maximally, cancel each other or anything


FIGURE 7.5: Results for Case A at location $x=20.101, y=0.525$ and $z=0.025$ for $l=\pi$, $\omega_{0}=1$ and $a=0$. Analytic approximation for $p_{1}$ is obtained from applying the method of stationary phase.


Figure 7.6: Magnifi cation of Fig.(7.6) for $t \in[24,40]$.
in between.
An extreme example of the influence of the length of the vibrating wall is presented in


Figure 7.7: Contour plots in the plane $y=\frac{1}{2}$ for the pressure perturbation at dimensionless times $t=10,20,30$ and 40. Results have been obtained for Case $A$ with $l=\pi, \omega_{0}=1$ and $a=0$.

Fig.(7.5). Fig.(7.5) shows the result for $a=0, \omega_{0}=1$ and $l=\pi$. For this specific problem the computational domain is given by $x \in[0,13 \pi], y \in[0,1]$ and $z \in[0,1]$, hence $x_{\max }=$ $13 \pi \approx 40.84$. The background mesh dimensions are given by $N_{x}=208, N_{y}=5$ and $N_{z}=5$, and the mesh contains 62400 tetrahedral elements.

For this specific choice of the parameters the solution for $p_{0}$, see Fig.(7.5), only shows one major "pulse" of width $2 \pi$. As explained, the solution for $p_{0}$ consists of a contribution from the near and from the far end of the vibrating wall. For $a=0$ and $l=\pi$ these two contributions cancel each other for $t \geq x+l$ (independent of $\omega_{0}$ ). In the figure the "pulse" is given by the contribution from the near end of the vibrating wall, as soon as the contribution from the far end of the vibrating wall reaches the microphone position $(t=x+l)$ the two signals cancel each other. From this time on only the contribution to the solution from the $p_{d}$ is visible. This specific choice of parameters provides a perfect opportunity to compare the numerical solution for $p=p_{d}$ with the analytical approximation for $p_{1}$ given by the method of stationary phase, Eq.(6.119).

From Fig.(7.5) it is observed that the analytical approximation to $p=p_{0}+p_{d}$ shows very good agreement with the numerical result. In Fig.(7.6) a magnification of the results for $t \in[24,40]$ is shown. The solution $p_{0}$ is now zero. The analytic approximation obtained for $p_{d}$, given by Eq.(6.119), shows good agreement with the numerical result. The analytical approximation slightly under-predicts the amplitude, but the frequency is well resolved. The analytical approximation has been evaluated numerically, where the summation over $k$ has been truncated at a certain large number of $k$ (of the order of $k=1000$ ). Using a larger value for $k$ did not significantly influence the solution.

In Fig.(7.7) contour plots in the plane $y=\frac{1}{2}$ for the pressure perturbation at dimensionless times $t=10,20,30$ and 40 are presented. In Fig.(7.7) only pressure levels between -0.01 and 0.01 have been presented. In Fig.(7.7) the pressure perturbation given by $p_{0}$ is presented by the black band of width $2 \pi$ which moves to the right with increasing time. At these pressure levels it is clearly observed that the pressure fluctuations given by $p_{d}$ form an antisymmetric pattern with respect to $z=\frac{1}{2}$ in the duct. It is furthermore observed that at greater distances from the vibrating wall the amplitude of the pressure perturbations given by $p_{d}$ decreases.

The only parameter which has not been discussed so far is the parameter $a$, which provides damping to the wall vibration. It has been verified that the solution shows the exact same damping factor as the imposed wall vibration, i.e. $e^{-a t}$.

In conclusion the obtained results for Case A have been shown to be in very good agreement with the analytic solution $p_{0}$ and analytic approximation $p_{d}$. The solution $p_{0}$ is observed to be a propagation mode (when $a=0$ ), $p_{d}$ represents (slowly) decaying modes which may still be observed at relatively large distances from the vibrating wall.

### 7.3 Results case B

In this section numerical results for the problem described in section 6.4.2 are presented and compared with the analytical solution, which has been presented in section 6.4.2 as well. In the problem of Case B, there are a three parameters which may be varied, i.e. the length of the vibrating part of the wall $2 l$, the frequency of the vibration $\omega_{0}$ and the frequency of the


Figure 7.8: Results for Case B at location $x=20.075, y=0.525$ and $z=0.025$ for $l=1$, $\omega_{0}=1$ and $\chi_{n}=\frac{\pi}{2}$.


Figure 7.9: Same as Fig.(7.8) but for $\chi_{n}=\frac{3 \pi}{2}$.
wall motion $\chi_{n}=\frac{n \pi}{2 l}$ ( $n$ odd), see also Eq.(6.122).
The solution for Case B has been observed to depend on the parameters $\omega_{0}$ and $l$ in a similar fashion as the solution of Case A, described in the previous section. Also the contribution of $p_{d}^{b}$ to the solution for $p^{b}$ shows many similarities with that of Case A. The influence of $p_{d}^{b}$
is largest close to the vibrating wall and becomes smaller and smaller when moving further down stream from the vibrating wall. As observed for Case A, $p_{d}^{b}$ represents (slowly) decaying modes which may still be observed at relatively large distances from the vibrating wall. The solution $p_{0}^{b}$ is a propagating mode.


Figure 7.10: Numerical result for Case B at location $x=10.0384, y=0.475$ and $z=$ 0.725 for $l=\frac{3}{2} \pi, \omega_{0}=1$ and $\chi_{n}=\frac{1}{3}$.

In Fig.(7.8) results for Case B for $l=1, \omega_{0}=1$ and $\chi_{n}=\frac{\pi}{2}$, so $n=1$, at microphone location $x=20.075, y=0.525$ and $z=0.025$ are presented. The results, presented in Fig.(7.8), have been obtained for a computational domain given by $x \in[0,40], y \in[0,1]$ and $z \in[0,1]$. The computational domain has been partitioned into a tetrahedral mesh consisting of 60000 elements. As mentioned in the introduction of this chapter, the domain is first divided into equally sized cubes, subsequently each cube is divided into twelve identical tetrahedrons. The cubical background mesh dimensions are given by $N_{x}=200, N_{y}=5$ and $N_{z}=5$.

In Fig.(7.9) results for the same problem at the same location are presented, however this time $\chi_{n}=\frac{3 \pi}{2}$. For $\chi_{n}=\frac{\pi}{2}$ the result is observed to be in very good agreement with the analytical solution $p_{0}^{b}$, given by Eq.(6.126). The results, presented in Fig.(7.9), have been obtained for a computational domain given by $x \in[0,40], y \in[0,1]$ and $z \in[0,1]$, which was divided into a mesh of 480000 elements. Results have also been obtained for this problem on a tetrahedral mesh consisting of 60000 elements. The results obtained on the two meshes showed only very small differences, which are too small to be noticed on the plotting scale of Fig.(7.9).

For $\chi_{n}=\frac{3 \pi}{2}$ a slight difference between the numeric result and $p_{0}^{b}$ can be observed. This is thought to be caused by the contribution of $p_{d}^{b}$, which has not been included in the


Figure 7.11: Results for Case B at location $x=20.0522, y=0.475$ and $z=0.725$ for $l=\frac{3}{2} \pi, \omega_{0}=1$ and $\chi_{n}=\frac{1}{3}$. Analytical approximation for $p_{1}^{b}$ is obtained from applying the method of stationary phase.


Figure 7.12: Magnifi cation of Fig.(7.11) for $t \in[25,60]$.
presented analytical solution. For $\chi_{n}=\frac{5 \pi}{2}$ (not shown) the observed difference between the numeric result and $p_{0}^{b}$ increased a little. Especially in the first dimensionless time units
after the signal has reached the microphone, rapid oscillations around $p_{0}^{b}$ are observed. For $\chi_{n}=\frac{5 \pi}{2}$ these oscillations are more rapid than for $\chi_{n}=\frac{3 \pi}{2}$. With increasing $\chi_{n}=$ it is furthermore observed that the amplitude of the pressure perturbation dicreases, compare also the amplitude of the results presented in Fig.(7.8) and Fig.(7.9).

As for Case A, there is a special length of the vibrating wall for which the solution $p_{0}^{b}$ cancels for $t \geq x+l$. For Case B this length is given by $l=\frac{3}{2} \pi$. Again this special situation can be used to compare the numerical result with the analytical approximation obtained from employing the method of stationary phase. In section 6.4.2 also the solution for $p_{d}^{b}$ was derived for $t \rightarrow \infty$. In Fig.(7.10) the numerical result is presented at microphone location $x=10.0384, y=0.475$ and $z=0.725$ for $l=\frac{3}{2} \pi, \omega_{0}=1$ and $\chi_{n}=\frac{1}{3}$. For the numerical simulation the computational domain was extended to $x_{\max }=64 \pi \approx 201$, see also the introduction of this chapter for a description of $x_{\max }$, resulting in a grid of 307,200 tetrahedral elements ( $N_{x}=1024, N_{y}=5$ and $N_{z}=5$ ). Usually the computation is stopped at dimensionless time $t=40$, this time it was stopped at $t=300$. Unfortunately $t=300$ has proven not to be large enough to verify the analytic solution for $p_{1}^{b}$ for $t \rightarrow \infty$.

Fig.(7.11) shows the numerical result for Case B at location $x=20.0522, y=0.475$ and $z=0.725$ for $l=\frac{3}{2} \pi, \omega_{0}=1$ and $\chi_{n}=\frac{1}{3}$. Additionally in the figure the analytic approximation has been presented. From Fig.(7.11) the difference between the analytic and numerical results is hardly visible. From Fig.(7.12), which shows a magnification of the results for $t \in[25,60]$, the difference can be observed. Like observed for Case A, the approximation obtained from applying the method of stationary phase slightly underpredicts the amplitude, but the frequency shows very good agreement.

In conclusion, the results obtained for Case B have been found to be in very good agreement with the analytical solution presented in section 6.4.2.

### 7.4 Results case A with mean fbw

### 7.4.1 Uniform mean flow

Let us now consider the results obtained for Case A, for the case that there is a uniform mean flow present in the duct. The mean flow velocity vector, expressed as Mach-number components, is given by $(M, 0,0)^{T}$. In section 6.5.1 the analytical solution for $p_{0}$ has been presented.


Figure 7.13: Numerical results for Case A for four different Mach-numbers, showing influence of Mach-number on time at which first signal is observed at location $x=20.025$, $y=0.525$ and $z=0.475$. Parameters: $\omega_{0}=1, l=1$ and $a=0$.

The results which will be shown in this section have all been obtained for a computational domain with $x_{\max }=40$, i.e the computational domain is given by $x \in[-40,40], y \in[0,1]$ and $z \in[0,1]$. In all cases the computational domain has been partitioned into tetrahedral meshes consisting of 120000 elements.

Fig.(7.13) shows numerical results for four different Mach-numbers, i.e. $M=0.05,0.1$ 0.2 and 0.4 , at location $x=20.025, y=0.525$ and $z=0.475$. The computational domain considered is given by: $x \in[-40,40], y \in[0,1]$ and $z \in[0,1]$. The computational domain was partitioned into a cubical background with dimensions $N_{x}=400, N_{y}=5$ and $N_{z}=5$, resulting in a tetrahedral mesh 120000 of elements.

From Fig.(7.13) it is clear that with increased mean flow velocity $M$, the signal from the vibrating wall reaches the microphone earlier, as expected and also predicted by the analytic solution $p_{0}$, Eq.(6.176). Fig.(7.14) shows the numerical result and analytical solution $p_{0}$ for $M=0.1$ at microphone location $x=20.025, y=0.525$ and $z=0.475$. The numerical and analytical result cannot be distinguished from one other. In addition, the analytical solution


Figure 7.14: Numerical result for Case $A$ for $M=0.1$ at location $x=20.025, y=0.525$ and $z=0.475$, and analytic solution $p_{0}$ for $M=0$ and $M=0.1$ at same location. Parameters: $\omega_{0}=1, l=1$ and $a=0$.
$p_{0}$ for $M=0$ at the same microphone location is shown. It is observed that for $M>0$ not only the signal is received earlier than for $M=0$, also the amplitude has decreased. The observed frequency of the signal, however, is not changed.

In Fig.(7.15) the numerical result and analytical solution $p_{0}$ at location $x=-20.025$, $y=0.475$ and $z=0.525$ for $M=0.1$ and $\omega_{0}=1, l=1$ and $a=0$ are presented. The analytic solution for $p_{0}$ for $x<-l$ has also been presented in section 6.5.1. In Fig.(7.15), as reference, also the analytic solution $p_{0}$ has been presented for $M=0$. As expected, at a microphone location at a certain upstream distance from the vibrating wall, the signal is observed later than in the case $M=0$. From the figure it is furthermore observed that for $M>0$ the measured pressure amplitude is larger upstream than for $M=0$. The observed frequency of the signal, however, is not changed.

Although not many parameters have been varied for Case A with mean flow, it has been shown convincingly that also for the case including mean flow the numerical results and the analytical prediction of $p_{0}$ are in very good agreement.

### 7.4.2 Hagen-Poseuille mean flow for rectangular duct

For Case A, described in section 6.4.1, also a more realistic mean flow has been considered, namely a Hagen-Poiseuille flow. White [85] presents the exact solution for a Hagen-Poseuille flow in a rectangular duct. For a more elaborate description the reader is referred to [85].

Assume that there is a fully-developed incompressible mean flow in the duct, which is produced by a constant pressure drop $\frac{\partial p}{\partial x}$ in axial direction. For this condition the velocity becomes purely axial, i.e. the mean flow velocity components in $y$ and $z$-direction are zero and the mean flow velocity component in $x$-direction becomes, expressed in terms of a Mach



Figure 7.15: Numerical result for Case $A$ for $M=0.1$ at location $x=-20.025, y=0.475$ and $z=0.525$, and analytical solution $p_{0}$ for $M=0$ and $M=0.1$ at same location.
number, a function of $y$ and $z, M(y, z)$. For these conditions, the Navier-Stokes equations greatly simplify under these conditions.

In White [85] the dimensionfull velocity profile for a Hagen-Poiseuille flow in a rectangular section $-a \leq y^{*} \leq a,-b \leq z^{*} \leq b$ is given by:

$$
\begin{align*}
U\left(y^{*}, z^{*}\right)= & \frac{16 a^{2}}{\mu \pi^{3}}\left(-\frac{\partial P}{\partial x}\right) \sum_{m=1,3,5, \ldots}^{\infty}(-1)^{(m-1) / 2}\left[1-\frac{\cosh \left(m \pi z^{*} / 2\right)}{\cosh (m \pi / 2)}\right] \\
& \times \frac{\cos \left(m \pi y^{*} / 2\right)}{m^{3}} \tag{7.3}
\end{align*}
$$

and the flow rate by:

$$
\begin{equation*}
Q=\frac{4 b a^{3}}{3 \mu}\left(-\frac{\partial P}{\partial x}\right)\left[1-\frac{192 a}{\pi^{5} b \mu} \sum_{m=1,3,5, \ldots}^{\infty} \frac{\tanh (m \pi / 2)}{m \pi}\right] \tag{7.4}
\end{equation*}
$$

In dimensionless form the (axial) velocity profile (Mach number), for a square section $y, z \in$ $[0,1]$, is given by:

$$
\begin{align*}
M(y, z)= & \frac{16}{\pi^{3}} \frac{\operatorname{Re}}{M_{0}}\left(-\frac{\partial p_{0}}{\partial x}\right) \sum_{m=1,3,5, \ldots}^{\infty}(-1)^{(m-1) / 2}\left[1-\frac{\cosh (m \pi(2 z-1) / 2)}{\cosh (m \pi / 2)}\right] \\
& \times \frac{\cos (m \pi(2 y-1) / 2)}{m^{3}} \tag{7.5}
\end{align*}
$$

The flow rate is:

$$
\begin{align*}
Q & =\frac{128}{\pi^{4}} \frac{\mathrm{Re}}{M_{0}}\left(-\frac{\partial p_{0}}{\partial x}\right) \sum_{m=1,3,5, \ldots}^{\infty} \frac{1}{m^{4}}\left[1-\frac{2 \tanh (m \pi / 2)}{m \pi}\right] \\
Q & \approx-6.27638 \frac{\mathrm{Re}}{M_{0}}\left(-\frac{\partial p_{0}}{\partial x}\right) \tag{7.6}
\end{align*}
$$

where Re is the Reynolds number:

$$
\begin{equation*}
\operatorname{Re}=\frac{\rho_{0} U_{0} h}{\mu_{0}} \tag{7.7}
\end{equation*}
$$

$\rho_{0}$ is the mean flow density, $U_{0}$ is the $x$-component of the mean flow velocity at the centerline, $h$ is the height of the duct and $\mu$ is the viscosity. Furthermore, $M_{0}$ is the mean flow Machnumber:

$$
\begin{equation*}
M_{0}=\frac{U_{0}}{c_{0}} \tag{7.8}
\end{equation*}
$$

and $c_{0}$ is the speed of sound. The laminar flow assumption is assumed valid for $\operatorname{Re}<2000$. For air, a low Reynolds number often leads to a restriction on the maximum allowable mean flow velocity $U_{0}$, i.e. $U_{0}$ has to be relatively (very) small. Unfortunately also the influence of the mean flow on the propagation of sound is then very small. However, if we assume the density to be very small the mean flow velocity $U_{0}$ can be relatively high while the Reynolds number remains below 2000. During the re-entry of the space-shuttle for example, laminar flow conditions occur in which the density is very small (medium is still a continuum). Instead of a very low density the height of the duct can be chosen very small, resulting in a micro-channel.

In this section we will compare results obtained for the Hagen-Poiseuille mean flow with results obtained for a uniform mean flow, where in both cases the flow rate is the same. In both cases, the results which will be shown have been obtained for a computational domain with $x_{\max }=40$, i.e the computational domain is given by $x \in[-40,40], y \in[0,1]$ and $z \in[0,1]$. In both cases the mesh consists of 960000 elements.

Fig.(7.16) and Fig.(7.17) show the numerical results for Case A with uniform mean flow and with Hagen-Poiseuille mean flow for $\omega_{0}=1, l=1$ and $a=0$, at two different microphone locations. In Fig.(7.16) results are presented for a microphone which is located just above the vibrating part of the wall wall, where the differences between the results for the case with uniform mean flow and the case with Hagen-Poiseuille mean flow are largest. This close to the wall the mean flow velocity is almost zero in the case of the Hagen-Poiseuille mean flow. In the preceding section it was observed that the amplitude decreases with increasing mean flow velocity. In Fig.(7.16) the higher amplitude observed for the Hagen-Poiseuille mean flow case can therefore be explained. The results presented in Fig.(7.17) are obtained for a microphone which is located approximately 4.5 plate-lengths ( $2 l$ ) downstream of downstream edge of vibrating part of the wall. Again the amplitude for the case with uniform mean flow is lower.

Because the vibrating wall perturbs the flow in a region where the mean flow is almost zero in the case of the Hagen-Poiseuille mean flow, the perturbation spreads less fast than in the case of the uniform mean flow, in which the mean flow velocity is not zero near the wall. This also means that further downstream of the vibrating wall the signal will be received later for



Figure 7.16: Comparison of numerical results for Case $A$ with uniform mean flow and with Hagen-Poiseuille mean flow. Uniform mean flow velocity is $M=0.2$, microphone location $x=0.5625, y=0.5125$ and $z=0.0375$. Parameters: $\omega_{0}=1, l=1$ and $a=0$.
the Hagen-Poiseuille mean flow case than for the uniform mean flow case. Fig.(7.17) shows that, indeed, the signal for the Hagen-Poiseuille mean flow case is received significantly later than for the uniform mean flow case.

### 7.5 Results case B with mean fbw

The results which will be shown in this section have all been obtained for a computational domain given by $x \in[-40,40], y \in[0,1]$ and $z \in[0,1]$. In all cases the computational domain has been partitioned into a mesh consisting of 120000 tetrahedral elements.

In Fig.(7.18) the numerical results for $M=0.1$ at microphone locations $(x, y, z)=$ $(20.025,0.525,0.475)$ and $(-20.025,0.475,0.525)$ are presented.

In addition, the analytical solution $p_{0}$ for $M=0.1$ and $M=0$, obtained for $x=20.025$, are presented in Fig.(7.18). The two microphone locations are almost exactly symmetric with respect to the plane $x=0$. From Fig.(7.18) it is observed that the analytical solution $p_{0}$, given by Eq.(6.188), shows perfect agreement with the numerical result at the same microphone location. Secondly it is observed that, with the result for $M=0$ as reference, the amplitude of the pressure is larger downstream and smaller up stream of $x=0$. The same behavior was observed for Case A with mean flow, presented in the preceding section.

Also for Case B with mean flow, the numerical results and analytical solution have been observed to show very good agreement.


Figure 7.17: Comparison of numerical results for Case A with uniform mean flow and with Hagen-Poiseuille mean flow. Uniform mean flow velocity is $M=0.2$, microphone location $x=10.0625, y=0.5125$ and $z=0.5375$. Parameters: $\omega_{0}=1, l=1$ and $a=0$.


FIGURE 7.18: Numerical result and analytical solution p0 at location $x=20.025, y=0.525$ and $z=0.475$ for Case $B$ with $M=0.1$.

## CONCLUSIONS AND RECOMMENDATIONS

### 8.1 Conclusion

In this thesis a numerical method has been presented for the accurate simulation of the propagation of acoustic information through fluids moving with non-uniform velocity in threedimensional complex domains.

It has been shown in chapter 2 that the propagation of (aero-) acoustic information through inhomogeneous moving fluids can be described by the linearized Euler equations, assuming that there is no feedback from the acoustic field to the back ground flow. The developed numerical method solves the linearized Euler equations for the primitive perturbation variables on tetrahedral elements employing the Discontinuous Galerkin method for the spatial discretization and the multi-stage low-storage Runge-Kutta method for the time discretization.

The discontinuous Galerkin method (chapter 3) has been chosen (for the spatial discretization) because it can be applied to formally obtain higher order accuracy on unstructured meshes. Computational methods used in aeroacoustics (CAA) are often higher order methods because it is of utmost importance to minimize numerical dissipation and dispersion.

In literature, $\mathrm{Hu} \&$ Atkins [42] report convergence rates of $h^{2 p+1}$ and $h^{2 p+2}$ for the dissipation and dispersion errors, respectively, where $h$ is the uniform element mesh size and $p$ the highest polynomial degree. By means of an extensive one-dimensional wave-propagation analysis (chapter 4) it has been shown that the dissipation and dispersion errors, obtained for the most accurate mode, converge at a rate of $h^{2 p+2}$ and $h^{2 p+3}$. Because of these high convergence rates of the dissipation and dispersion errors, the discontinuous Galerkin method is very well suited for application to acoustic propagation problems.

In this thesis two verification problems have been considered: the convection of a twodimensional Gaussian pulse in a uniform mean flow and the acoustic radiation from a vibrating wall segment inside an infinite rectangular duct. For both problems, the results have been obtained while using polynomial basis functions in the discontinuous Galerkin method of degree $\leq 1$. In the low-storage Runge-Kutta time integration method four stages have been used, with the coefficients chosen such that the time integration is fourth-order accurate in terms of the truncation error.

The first verification problem is presented in chapter 5.The results obtained are shown to be in good agreement with the analytical solution. It is furthermore shown that the obtained numerical solution converges at a rate at least $h^{2 \frac{1}{2}}$, meassured in a norm defined employing all common node points of meshes considered (ranging from coarse to fine). In addition, a
performance test of the parallelized algorithm implemented on the super computer TERAS, showed that (near) linear speed-up is obtained for up to 32 processors on a mesh of approximately 0.6 million tetrahedral elements.

For the second verification problem the analytical solution has been obtained (chapter 6). To the author's knowledge the analytical solution for this specific problem has not been presented before in literature. The obtained numerical solutions have been compared with the analytical solutions and shown to be in good agreement (chaper 7).

### 8.2 Recommendations

In chapter 3 the discontinuous Galerkin method is applied to a simplified form of the linearized Euler equations, i.e. the terms $D_{0} \mathbf{q}^{\prime}$ and $Y_{0} \mathbf{q}^{\prime}$ in Eq.(2.138) have been neglected. When there are no sources in the background flow, which is a common assumption in acoustics, the matrix $Y_{0}=0$. In the considered verification problems the background flow has been assumed uniform and, because the matrix is a function of the mean-flow derivatives (see Eq.(2.140)), $D_{0}=0$. In section 7.4.2, however, the mean flow was given by a HagenPoiseuille mean flow and some components of the matrix $D_{0}$ are no longer zero. From the point of view of applications of the numerical method to more general, complex flows, it is strongly recommended to investigate the effects of including the term $D_{0} \mathbf{q}^{\prime}$.

Development of, so-called, proper non-reflecting boundary conditions is an important subject in the field of computational aeroacoustics. In this thesis only characteristic-based nonreflecting boundary conditions have been applied. It is known from literature that these boundary conditions may result in numerics-induced reflections which may contaminate the solution at later times. In the literature, other boundary conditions such as, so-called, sponge layers and Perfectly Matched Layers (PML) have been considered. In the present study, the sponge layer boundary condition has also been implemented, however, not yet been verified. It is recommended to analyze the performance of these boundary conditions, also because they are of great importance for application involving complex flows.

In chapter 4 an extensive investigation has been presented of the wave-propagation properties of the semi-discrete system, obtained from applying the discontinuous Galerkin method for the spatial discretization to the one-dimensional scalar advection equation. For the fullydiscrete system only the stability has been considered. It is known from literature that in most cases the maximum allowable time step (for a given mesh) is determined by the required accuracy and not by the stability. It is recommended to investigate the dissipation and dispersion errors of the fully discrete system in a similar way as has been considered for the semi-discrete system. It would be interesting to investigate if it is possible to optimize the coefficients of the low-storage multi-stage Runge-Kutta method to minimize dispersion and dissipation like is pursued in the DRP-schemes of Tam \& Webb (Tam \& Webb [75], Tam [76], [77]).

Furthermore, the wave-propagation analysis should be extended to two spatial dimensions. To the author's knowledge only Hu et al. ([43]) have investigated the wave-propagation properties of the dicontinuous Galerkin method in two spatial dimensions. Their analysis has been limited to $p \leq 6$. One of their conclusions is that the orientation of elements in a mesh introduces anisotropy in the phase speed (dispersion) as well as the damping rate (dissipation), which is an important finding.

In this thesis the generated output of the computational method has been visualized employing a commercial package, which required data at the nodes. In the DG method this implies that we have to average the solutions in a node which are given by all the elements which have the particular node in common. For detailed study of the results of the method it is recommended to preserve the expansion coefficients of the formulation.

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## EULER EQUATION MANIPULATIONS

In the first section of this appendix the derivation of the flux Jacobians for the Euler equations, which are used in chapter 2, is presented. In the second section it is shown how the Euler equations can be converted from conservative variables to primitive variables.

## A. 1 Derivation of flux Jacobians for the Euler equations

The flux Jacobians:

$$
\begin{equation*}
B_{j} \equiv \frac{d \mathbf{F}_{j}}{d \mathbf{U}}(\mathbf{U}) \tag{1.1}
\end{equation*}
$$

can be obtained conveniently when we use the notation:

$$
\mathbf{U}=\left(\begin{array}{c}
\rho  \tag{1.2}\\
\rho u \\
\rho v \\
\rho w \\
\rho E
\end{array}\right)=\left(\begin{array}{c}
w_{1} \\
w_{2} \\
w_{3} \\
w_{4} \\
w_{5}
\end{array}\right) .
$$

Subsequently we can write the fluxes $\mathbf{F}_{j}$ results in terms of the $w_{k}$ 's:

$$
\begin{align*}
\mathbf{F}_{j}(\mathbf{U}) & =\left(\begin{array}{c}
\rho u_{j} \\
\rho u u_{j}+\delta_{1 j} p \\
\rho v u_{j}+\delta_{2 j} p \\
\rho w u_{j}+\delta_{3 j} p \\
\rho H u_{j}
\end{array}\right) \\
& =\left(\begin{array}{c}
w_{1+j} \\
\frac{w_{2} w_{1+j}}{w_{1}}+\delta_{1 j}(\gamma-1)\left[\begin{array}{c}
w_{5}-\frac{1}{2 w_{1}}\left(w_{2}^{2}+w_{3}^{2}+w_{4}^{2}\right) \\
\frac{w_{3} w_{1+j}}{w_{1}}+\delta_{2 j}(\gamma-1) \\
\frac{w_{4} w_{1+j}}{w_{1}}+\delta_{3 j}(\gamma-1)\left[\begin{array}{c}
w_{5}-\frac{1}{2 w_{1}}\left(w_{2}^{2}+w_{3}^{2}+w_{4}^{2}\right) \\
\frac{w_{1+j}}{w_{1}}\left[\gamma w_{5}-\frac{1}{2 w_{1}}\left(w_{2}^{2}+w_{3}^{2}+w_{4}^{2}\right)\right.
\end{array}\right]
\end{array}\right) .
\end{array} . . \begin{array}{l}
\left.\frac{\gamma-1}{2 w_{1}}\left(w_{2}^{2}+w_{3}^{2}+w_{4}^{2}\right)\right]
\end{array}\right) . \tag{1.3}
\end{align*}
$$

The matrices $B_{j}$ are now obtained by differentiation to $w_{k}$.

For example for $j=1$ (x-direction) we have:

$$
\mathbf{F}_{1}=\left(\begin{array}{c}
w_{2}  \tag{1.4}\\
\frac{w_{2}^{2}}{w_{1}}+(\gamma-1)\left[w_{5}-\frac{1}{2 w_{1}}\left(w_{2}^{2}+w_{3}^{2}+w_{4}^{2}\right)\right] \\
\frac{w_{2} w_{3}}{w_{1}} \\
\frac{w_{2} w_{4}}{w_{1}} \\
\frac{w_{2}}{w_{1}}\left[\gamma w_{5}-\frac{\gamma-1}{2 w_{1}}\left(w_{2}^{2}+w_{3}^{2}+w_{4}^{2}\right)\right]
\end{array}\right)
$$

and, with $F_{1 m}$ denoting the $m^{t h}$-element of vector $\mathbf{F}_{1}$ :

$$
B_{1}=\left[\begin{array}{lllll}
\frac{\partial F_{11}}{\partial w_{1}} & \frac{\partial F_{11}}{\partial w_{2}} & \frac{\partial F_{11}}{\partial w_{3}} & \frac{\partial F_{11}}{\partial w_{4}} & \frac{\partial F_{11}}{\partial w_{5}}  \tag{1.5}\\
\frac{\partial F_{12}}{\partial w_{1}} & \frac{\partial F_{12}}{\partial w_{2}} & \frac{\partial F_{12}}{\partial w_{3}} & \frac{\partial F_{12}}{\partial w_{4}} & \frac{\partial F_{12}}{\partial w_{5}} \\
\frac{\partial F_{13}}{\partial w_{1}} & \frac{\partial F_{13}}{\partial w_{2}} & \frac{\partial F_{13}}{\partial w_{3}} & \frac{\partial F_{13}}{\partial w_{4}} & \frac{\partial F_{13}}{\partial w_{5}} \\
\frac{\partial F_{14}}{\partial w_{1}} & \frac{\partial F_{14}}{\partial w_{2}} & \frac{\partial F_{14}}{\partial w_{3}} & \frac{\partial F_{14}}{\partial w_{4}} & \frac{\partial F_{14}}{\partial w_{5}} \\
\frac{\partial F_{15}}{\partial w_{1}} & \frac{\partial F_{15}}{\partial w_{2}} & \frac{\partial F_{15}}{\partial w_{3}} & \frac{\partial F_{15}}{\partial w_{4}} & \frac{\partial F_{15}}{\partial w_{5}}
\end{array}\right]
$$

$$
=\left[\begin{array}{ccccc}
0 & 1 & 0 & 0 & 0  \tag{1.6}\\
-\frac{w_{2}^{2}}{w_{1}^{2}}+\left(\gamma-1^{2}\right) \frac{1}{2 w_{1}^{2}}|\vec{w}|^{2} & (3-\gamma) \frac{w_{2}}{w_{1}} & -(\gamma-1) \frac{w_{3}}{w_{1}} & -(\gamma-1) \frac{w_{4}}{w_{1}} & \gamma-1 \\
-\frac{w_{2} w_{3}}{w_{2}^{2}} & \frac{w_{3}}{w_{1}} & \frac{w_{2}}{w_{1}} & 0 & 0 \\
-\frac{w_{2} w_{4}^{2}}{w_{1}^{2}} & \frac{w_{4}}{w_{1}} & 0 & \frac{w_{2}}{w_{1}} & 0 \\
-\gamma \frac{w_{2} w_{5}}{w_{1}^{2}}+(\gamma-1) \frac{w_{2}}{w_{1}^{3}}|\vec{w}|^{2} & \gamma \frac{w_{5}}{w_{1}}-\frac{\gamma-1}{2 w_{1}^{2}}\left(|\vec{w}|^{2}+2 w_{2}^{2}\right) & -(\gamma-1) \frac{w_{2} w_{3}}{w_{1}^{2}} & -(\gamma-1) \frac{w_{2} w_{4}}{w_{1}^{2}} & \gamma \frac{w_{2}}{w_{1}}
\end{array}\right]
$$

where we used the notation $|\vec{w}|^{2}=w_{2}^{2}+w_{3}^{2}+w_{4}^{2}$. We obtain, after replacing the $w$ 's by the proper conservative variables, in x-direction:
$B_{1}=\left[\begin{array}{ccccc}0 & 1 & 0 & 0 & 0 \\ -u^{2}+\frac{(\gamma-1)}{2}|\vec{u}|^{2} & (3-\gamma) u & -(\gamma-1) v & -(\gamma-1) w & (\gamma-1) \\ -u v & v & u & 0 & 0 \\ -u w & w & 0 & u & 0 \\ u\left[-\gamma E+(\gamma-1)|\vec{u}|^{2}\right] & \gamma E-\frac{(\gamma-1)}{2}\left(|\vec{u}|^{2}+2 u^{2}\right) & -(\gamma-1) u v & -(\gamma-1) u w & \gamma u\end{array}\right]$.
The matrices $B_{2}$ and $B_{3}$ can be obtained analogously; they are given by:
$B_{2}=\left[\begin{array}{ccccc}0 & 0 & 1 & 0 & 0 \\ -u v & v & u & 0 & 0 \\ -v^{2}+\frac{(\gamma-1)}{2}|\vec{u}|^{2} & -(\gamma-1) u & (3-\gamma) v & -(\gamma-1) w & (\gamma-1) \\ -v w & 0 & w & v & 0 \\ v\left[-\gamma E+(\gamma-1)|\vec{u}|^{2}\right] & -(\gamma-1) u v & \gamma E-\frac{(\gamma-1)}{2}\left(|\vec{u}|^{2}+2 v^{2}\right) & -(\gamma-1) v w & \gamma v\end{array}\right]$,
and
$B_{3}=\left[\begin{array}{ccccc}0 & 0 & 0 & 1 & 0 \\ -u w & w & 0 & u & 0 \\ -v w & 0 & w & v & 0 \\ -w^{2}+\frac{(\gamma-1)}{2}|\vec{u}|^{2} & -(\gamma-1) u & -(\gamma-1) v & (3-\gamma) w & (\gamma-1) \\ w\left[-\gamma E+(\gamma-1)|\vec{u}|^{2}\right] & -(\gamma-1) u w & -(\gamma-1) v w & \gamma E-\frac{(\gamma-1)}{2}\left(|\vec{u}|^{2}+2 w^{2}\right) & \gamma w\end{array}\right]$.

## §A.2. Converting Euler Equations from conservative to primitive variables 169

## A. 2 Converting Euler Equations from conservative to primitive variables

In this appendix we will show how the Euler equations can be converted from conservative variables $(\rho, \rho u, \rho v, \rho w, \rho E)$ to primitive variables $(\rho, u, v, w, p)$.

The Euler equations for the conservative variables read:

$$
\begin{align*}
& \frac{\partial \rho}{\partial t}+\frac{\partial}{\partial x_{j}}\left(\rho u_{j}\right)=S_{m}  \tag{1.10}\\
& \frac{\partial \rho u_{i}}{\partial t}+\frac{\partial}{\partial x_{j}}\left(\rho u_{j} u_{i}+p \delta_{i j}\right)=S_{i},  \tag{1.11}\\
& \frac{\partial \rho E}{\partial t}+\frac{\partial}{\partial x_{j}}\left(\rho H u_{j}\right)=S_{e} . \tag{1.12}
\end{align*}
$$

- continuity equation:

Equation (1.10) can be considered as an equation for the primitive variable $\rho$ :

$$
\begin{equation*}
\frac{\partial \rho}{\partial t}+u_{j} \frac{\partial \rho}{\partial x_{j}}+\rho \frac{\partial u_{j}}{\partial x_{j}}=S_{m} \tag{1.13}
\end{equation*}
$$

- momentum equation:

Equation (1.11) can be written as:

$$
\begin{equation*}
u_{i}\left\{\frac{\partial \rho}{\partial t}+\frac{\partial}{\partial x_{j}}\left(\rho u_{j}\right)\right\}+\rho\left\{\frac{\partial u_{i}}{\partial t}+u_{j} \frac{\partial u_{i}}{\partial x_{j}}\right\}+\frac{\partial}{\partial x_{j}}\left(p \delta_{i j}\right)=S_{i} \quad i=1,2,3 . \tag{1.14}
\end{equation*}
$$

After substitution of the continuity equation and some rearranging, we obtain the following equation for the primitive variable $u_{i}$ :

$$
\begin{equation*}
\frac{\partial u_{i}}{\partial t}+u_{j} \frac{\partial u_{i}}{\partial x_{j}}+\frac{1}{\rho} \frac{\partial}{\partial x_{j}}\left(p \delta_{i j}\right)=\frac{1}{\rho}\left(S_{i}-u_{i} S_{m}\right) \quad i=1,2,3 . \tag{1.15}
\end{equation*}
$$

- energy equation:

With $H=E+p / \rho$, equation (1.12) can be written as:

$$
\begin{equation*}
\rho\left\{\frac{\partial E}{\partial t}+u_{j} \frac{\partial E}{\partial x_{j}}\right\}+\frac{\partial}{\partial x_{j}}\left(p u_{j}\right)+E\left\{\frac{\partial \rho}{\partial t}+\frac{\partial}{\partial x_{j}}\left(\rho u_{j}\right)\right\}=S_{e} . \tag{1.16}
\end{equation*}
$$

We can write this as the following equation for the total energy $E$ :

$$
\begin{equation*}
\frac{\partial E}{\partial t}+u_{j} \frac{\partial E}{\partial x_{j}}+\frac{1}{\rho} \frac{\partial}{\partial x_{j}}\left(p u_{j}\right)=\frac{1}{\rho}\left(S_{e}-E S_{m}\right) \tag{1.17}
\end{equation*}
$$

With $E=e+\frac{1}{2} u_{i} u_{i}$ we can write this as an equation for the specific internal energy $e$ :

$$
\begin{equation*}
\frac{\partial e}{\partial t}+u_{j} \frac{\partial e}{\partial x_{j}}+\frac{1}{\rho} \frac{\partial}{\partial x_{j}}\left(p u_{j}\right)-\frac{u_{i}}{\rho} \frac{\partial}{\partial x_{j}}\left(p \delta_{i j}\right)=\frac{1}{\rho}\left(S_{e}-E S_{m}\right)-\frac{u_{i}}{\rho}\left(S_{i}-u_{i} S_{m}\right), \tag{1.18}
\end{equation*}
$$

where

$$
\begin{equation*}
\frac{1}{\rho} \frac{\partial}{\partial x_{j}}\left(p u_{j}\right)-\frac{u_{i}}{\rho} \frac{\partial}{\partial x_{j}}\left(p \delta_{i j}\right)=\frac{p}{\rho} \frac{\partial u_{j}}{\partial x_{j}} \tag{1.19}
\end{equation*}
$$

Assuming air behaves as an ideal gas and assuming $c_{v}$ to be constant, we may use:

$$
\begin{equation*}
e=c_{v} T=\frac{1}{\gamma-1} \frac{p}{\rho} . \tag{1.20}
\end{equation*}
$$

Upon substitution we obtain:

$$
\begin{gather*}
\frac{1}{\gamma-1} \frac{1}{\rho}\left\{\frac{\partial p}{\partial t}+u_{j} \frac{\partial p}{\partial x_{j}}\right\}-\frac{1}{\gamma-1} \frac{p}{\rho^{2}}\left\{\frac{\partial \rho}{\partial t}+u_{j} \frac{\partial \rho}{\partial x_{j}}\right\}+\frac{p}{\rho} \frac{\partial u_{j}}{\partial x_{j}}= \\
\frac{1}{\rho}\left(S_{e}-E S_{m}\right)-\frac{u_{i}}{\rho}\left(S_{i}-u_{i} S_{m}\right) \tag{1.21}
\end{gather*}
$$

Finally we obtain:

$$
\begin{equation*}
\frac{\partial p}{\partial t}+u_{j} \frac{\partial p}{\partial x_{j}}+\gamma p \frac{\partial u_{j}}{\partial x_{j}}=(\gamma-1)\left\{S_{e}-u_{i} S_{i}+\frac{1}{2} u_{i} u_{i} S_{m}\right\} \tag{1.22}
\end{equation*}
$$

Equations (1.13), (1.15) and (1.22) form the Euler equations in primitive variables. For completeness we will repeat them:

$$
\begin{align*}
& \frac{\partial \rho}{\partial t}+u_{j} \frac{\partial \rho}{\partial x_{j}}+\rho \frac{\partial u_{j}}{\partial x_{j}}=S_{m} \\
& \frac{\partial u_{i}}{\partial t}+u_{j} \frac{\partial u_{i}}{\partial x_{j}}+\frac{1}{\rho} \frac{\partial p}{\partial x_{i}}=\frac{1}{\rho}\left(S_{i}-u_{i} S_{m}\right), \quad i=1,2,3 \\
& \frac{\partial p}{\partial t}+u_{j} \frac{\partial p}{\partial x_{j}}+\gamma p \frac{\partial u_{j}}{\partial x_{j}}=(\gamma-1)\left\{S_{e}-u_{i} S_{i}+\frac{1}{2} u_{i} u_{i} S_{m}\right\} \tag{1.23}
\end{align*}
$$

## Proof of Eigenvalue THEOREM

The characteristic polynomial that contains the eigenvalues of the matrix $-\bar{K}$ as its roots, is given by

$$
\begin{equation*}
\operatorname{det}(-\bar{K}-\lambda I)=0, \quad-\bar{K}=\bar{A}-\mu^{-1} \bar{B}-\epsilon\left[\mu^{-1} \bar{B}-\bar{C}+\mu \bar{B}^{T}\right] \tag{2.1}
\end{equation*}
$$

where $I$ is the $(p+1) \times(p+1)$-identity matrix. In this appendix we will proof that the characteristic polynomial we are looking for is given by:

$$
\begin{equation*}
\left[(\mu-1) \phi_{0}^{p}(\lambda, \mu ; \epsilon)+(\mu+1) \phi_{1}^{p}(\lambda, \mu ; \epsilon)-\mu\right] \lambda^{p+1}=0 \tag{2.2}
\end{equation*}
$$

with

$$
\begin{align*}
& \phi_{k}^{p}(\lambda, \mu ; \epsilon)=\phi_{k+2}^{p}(\lambda, \mu ; \epsilon)-\frac{2 k+1}{\lambda}\left[2 \phi_{k+1}^{p}(\lambda, \mu ; \epsilon)-1-\epsilon\left(1-(-1)^{k} \mu\right)\right], \quad k \in[0, p] \\
& \phi_{k}^{p}(\lambda, \mu ; \epsilon)=0, \quad k>p \tag{2.3}
\end{align*}
$$

We start by defining three $(p+1) \times(p+1)$-matrices $D, K_{1}$ and $K_{2}$ :

$$
\begin{align*}
D_{m k} & \equiv \begin{cases}\sqrt{2 m+1}, & m=k, \\
0, & \text { otherwise },\end{cases}  \tag{2.4}\\
\left(K_{1}\right)_{m k} & \equiv \begin{cases}1, & m \geq 0, \quad k=m, \\
\phi_{k}^{p}, & m=0, \quad k \geq 1, \\
0, & \text { otherwise },\end{cases}  \tag{2.5}\\
\left(K_{2}\right)_{m k} & \equiv \begin{cases}\frac{\mu}{\mu-1}, & m=0, \quad k=0, \\
-\frac{\mu+1}{\mu-1}, & m=1, \quad k=0 \\
-1, & m \geq 2, \quad k=m-2, \\
1, & m \geq 1, \quad k=m, \\
0, & \text { otherwise. }\end{cases} \tag{2.6}
\end{align*}
$$

Since $D, K_{1}$ and $K_{2}$ are a diagonal matrix, an upper triangular matrix and a lower triangular matrix, respectively, it is readily verified that

$$
\begin{equation*}
\operatorname{det}(D)=\prod_{m=0}^{p} \sqrt{2 m+1}, \quad \operatorname{det}\left(K_{1}\right)=1, \quad \operatorname{det}\left(K_{2}\right)=\frac{\mu}{\mu-1} \tag{2.7}
\end{equation*}
$$

Hence, since it will be assumed that $\mu \neq 1$, all of these matrices have finite non-zero determinants and therefore are invertible.

Next we combine matrices $K_{1}$ and $K_{2}$ :

$$
\begin{equation*}
K \equiv K_{1} K_{2} \tag{2.8}
\end{equation*}
$$

hence

$$
K_{m k} \equiv \begin{cases}\frac{\mu}{\mu-1}-\frac{\mu+1}{\mu-1} \phi_{1}^{p}-\phi_{2}^{p}, & m=0, \quad k=0  \tag{2.9}\\ \phi_{k}^{p}-\phi_{k+2}^{p}, & m=0, \quad k \geq 1, \\ -\frac{\mu+1}{\mu-1}, & m=1, \quad k=0, \\ 1, & m \geq 1, \quad k=m \\ -1, & m \geq 2, \quad k=m-2 \\ 0, & \text { otherwise } .\end{cases}
$$

Since

$$
\begin{equation*}
\operatorname{det}(K)=\operatorname{det}\left(K_{1}\right) \operatorname{det}\left(K_{2}\right)=\frac{\mu}{\mu-1} \tag{2.10}
\end{equation*}
$$

and $\mu \neq 1$ by assumption, also $K$ has a finite non-zero determinant and therefore is invertible.
We now write

$$
\begin{equation*}
-\bar{K}-\lambda I=D K^{-1}\left(K D^{-1}(-\bar{K}-\lambda I) D^{-1}\right) D \tag{2.11}
\end{equation*}
$$

and will show that $K D^{-1}(-\bar{K}-\lambda I) D^{-1}$ is a lower triangular matrix such that its determinant is simply the product of its diagonal elements.

It is easily verified that

$$
\begin{align*}
\left(D^{-1} \bar{A} D^{-1}\right)_{m k} & = \begin{cases}-1, & m>k, \quad m-k \\
1, & \text { otherwise },\end{cases}  \tag{2.12}\\
\left(D^{-1} \bar{B} D^{-1}\right)_{m k} & = \begin{cases}-1, & m \text { odd }, \\
1, & \text { otherwise },\end{cases}  \tag{2.13}\\
\left(D^{-1} \bar{B}^{T} D^{-1}\right)_{m k} & = \begin{cases}-1, & k \text { odd } \\
1, & \text { otherwise }\end{cases}  \tag{2.14}\\
\left(D^{-1} \bar{C} D^{-1}\right)_{m k} & = \begin{cases}2, & m+k \text { even } \\
0, & \text { otherwise }\end{cases} \tag{2.15}
\end{align*}
$$

and

$$
\left(D^{-1} I D^{-1}\right)_{m k}= \begin{cases}1 /(2 m+1), & m=k  \tag{2.16}\\ 0, & \text { otherwise }\end{cases}
$$

Next, we evaluate the matrices $K D^{-1} \bar{A} D^{-1}, K D^{-1} \bar{B} D^{-1}, K D^{-1} \bar{B}^{T} D^{-1}, K D^{-1} \bar{C} D^{-1}$
and $K D^{-1} I D^{-1}$, respectively. We start with $K D^{-1} \bar{A} D^{-1}$ :

$$
\begin{align*}
& \left(K D^{-1} \bar{A} D^{-1}\right)_{0 k}=\sum_{j=0}^{p} K_{0 j}\left(D^{-1} A D^{-1}\right)_{j k} \\
& =\left(\frac{\mu}{\mu-1}-\frac{\mu+1}{\mu-1} \phi_{1}^{p}-\phi_{2}^{p}\right)\left(D^{-1} \bar{A} D^{-1}\right)_{00} \\
& +\sum_{j=1}^{k} K_{0 j}+\sum_{j=k+1}^{p}(-1)^{j-k} K_{0 j} \\
& =\frac{\mu}{\mu-1}-\frac{\mu+1}{\mu-1} \phi_{1}^{p}-\phi_{2}^{p}  \tag{2.17}\\
& +\sum_{j=1}^{k}\left(\phi_{j}^{p}-\phi_{j+2}^{p}\right) \\
& +\sum_{j=k+1}^{p}(-1)^{j-k}\left(\phi_{j}^{p}-\phi_{j+2}^{p}\right), \\
& \left(K D^{-1} \bar{A} D^{-1}\right)_{10}=\sum_{j=0}^{p} K_{1 j}\left(D^{-1} \bar{A} D^{-1}\right)_{j 0} \\
& =-\frac{\mu+1}{\mu-1}\left(D^{-1} \bar{A} D^{-1}\right)_{00}+\left(D^{-1} \bar{A} D^{-1}\right)_{10}  \tag{2.18}\\
& =-\frac{2 \mu}{\mu-1} \text {, } \\
& \left.\left(K D^{-1} \bar{A} D^{-1}\right)_{1 k}\right|_{k \geq 1}=\left.\sum_{j=0}^{p} K_{1 j}\left(D^{-1} \bar{A} D^{-1}\right)_{j k}\right|_{k \geq 1} \\
& =-\left.\frac{1-\mu}{1+\mu}\left(D^{-1} \bar{A} D^{-1}\right)_{0 k}\right|_{k \geq 1}  \tag{2.19}\\
& +\left.\left(D^{-1} \bar{A} D^{-1}\right)_{1 k}\right|_{k \geq 1} \\
& =-\frac{2}{\mu-1} \text {, } \\
& \left.\left(K D^{-1} \bar{A} D^{-1}\right)_{m k}\right|_{m \geq 2}=\left.\sum_{j=0}^{p} K_{m j}\left(D^{-1} \bar{A} D^{-1}\right)_{j k}\right|_{m \geq 2} \\
& =\left.K_{m m}\left(D^{-1} \bar{A} D^{-1}\right)_{m k}\right|_{m \geq 2} \\
& +\left.K_{m m-2}\left(D^{-1} \bar{A} D^{-1}\right)_{m-2 k}\right|_{m \geq 2}  \tag{2.20}\\
& =\left.\left\{\left(D^{-1} \bar{A} D^{-1}\right)_{m k}-\left(D^{-1} \bar{A} D^{-1}\right)_{m-2 k}\right\}\right|_{m \geq 2} \\
& = \begin{cases}-2, & k=m-1 \\
0, & \text { otherwise } .\end{cases}
\end{align*}
$$

The last step in Eq. (2.20) is based on discriminating between the following possibilities for the value of $m$ :

$$
\begin{equation*}
m \leq k, \quad m=k+1, \quad m=k+2, \quad m>k+2 \tag{2.21}
\end{equation*}
$$

and noting that only in the case of $m=k+1$ the terms $\left(D^{-1} \bar{A} D^{-1}\right)_{m k}$ and $\left(D^{-1} \bar{A} D^{-1}\right)_{m-2 k}$ have opposite signs while in the other cases they have equal signs and cancel.

If we define functions $\Phi_{k}^{p}(\lambda)$ as:

$$
\begin{align*}
\Phi_{k}^{p}(\lambda) & \equiv \sum_{m=1}^{k}\left(\phi_{m}^{p}-\phi_{m+2}^{p}\right)+\sum_{m=k+1}^{p}(-1)^{m-k}\left(\phi_{m}^{p}-\phi_{m+2}^{p}\right) \\
& =\sum_{m=1}^{2} \phi_{m}^{p}+\sum_{m=k+1}^{k+2}\left\{(-1)^{m-k}-1\right\} \phi_{m}^{p} \tag{2.22}
\end{align*}
$$

we can summarize:

$$
\left(K D^{-1} \bar{A} D^{-1}\right)_{m k}= \begin{cases}\frac{\mu}{\mu-1}-\frac{\mu+1}{\mu-1} \phi_{1}^{p}-\phi_{2}^{p}+\Phi_{k}^{p}, & m=0  \tag{2.23}\\ -\frac{2 \mu}{\mu-1}, & m=1, \quad k=0 \\ -\frac{2}{\mu-1}, & m=1, \quad k \geq 1 \\ -2, & m \geq 2, k=m-1 \\ 0, & \text { otherwise } .\end{cases}
$$

We continue with $K D^{-1} \bar{B} D^{-1}$ :

$$
\begin{align*}
&\left(K D^{-1} \bar{B} D^{-1}\right)_{0 k}= \sum_{j=0}^{p} K_{0 j}\left(D^{-1} \bar{B} D^{-1}\right)_{j k} \\
&= \sum_{j=0}^{p}(-1)^{j} K_{0 j} \\
&= \frac{\mu}{\mu-1}-\frac{\mu+1}{\mu-1} \phi_{1}^{p}-\phi_{2}^{p}  \tag{2.24}\\
&+\sum_{j=1}^{p}(-1)^{j}\left(\phi_{j}^{p}-\phi_{j+2}^{p}\right) \\
&= \frac{\mu}{\mu-1}-\frac{\mu+1}{\mu-1} \phi_{1}^{p}-\phi_{1}^{p}=\frac{\mu}{\mu-1}\left(1-2 \phi_{1}^{p}\right), \\
&\left(K D^{-1} \bar{B} D^{-1}\right)_{1 k}= \sum_{j=0}^{p} K_{1 j}\left(D^{-1} \bar{B} D^{-1}\right)_{j k} \\
&= \sum_{j=0}^{p}(-1)^{j} K_{1 j}  \tag{2.25}\\
&=-\frac{\mu+1}{\mu-1}-1=-\frac{2}{\mu-1}, \\
&\left.\left(K D^{-1} \bar{B} D^{-1}\right)_{m k}\right|_{m \geq 2}=\left.\sum_{j=0}^{p} K_{m j}\left(D^{-1} \bar{B} D^{-1}\right)_{j k}\right|_{m \geq 2} \\
&=\left.\sum_{j=0}^{p}(-1)^{k} K_{m k}\right|_{m \geq 2}  \tag{2.26}\\
&=\left.\left\{K_{m m}(-1)^{m}+K_{m m-2}(-1)^{m-2}\right\}\right|_{m \geq 2} \\
&=\left.\left\{(-1)^{m}-(-1)^{m-2}\right\}\right|_{m>0}=0
\end{align*}
$$

In summary:

$$
\left(K D^{-1} \bar{B} D^{-1}\right)_{m k}= \begin{cases}\frac{\mu}{\mu-1}\left(1-2 \phi_{1}^{p}\right) & m=0  \tag{2.27}\\ -\frac{2 \mu}{\mu-1}, & m=1 \\ 0, & \text { otherwise }\end{cases}
$$

Next we evaluate $K D^{-1} \bar{B}^{T} D^{-1}$ :

$$
\begin{align*}
\left(K D^{-1} \bar{B}^{T} D^{-1}\right)_{0 k}= & \sum_{j=0}^{p} K_{0 j}\left(D^{-1} \bar{B}^{T} D^{-1}\right)_{j k} \\
= & (-1)^{k} \sum_{j=0}^{p} K_{0 j} \\
= & (-1)^{k}\left\{\frac{\mu}{\mu-1}-\frac{\mu+1}{\mu-1} \phi_{1}^{p}-\phi_{2}^{p}\right.  \tag{2.28}\\
& \left.+\sum_{j=1}^{p}\left(\phi_{j}^{p}-\phi_{j+2}^{p}\right)\right\} \\
= & (-1)^{k} \frac{\mu-2 \phi_{1}^{p}}{\mu-1}
\end{align*}
$$

$$
\begin{align*}
\left(K D^{-1} \bar{B}^{T} D^{-1}\right)_{1 k} & =\sum_{j=0}^{p} K_{1 j}\left(D^{-1} \bar{B}^{T} D^{-1}\right)_{j k} \\
& =(-1)^{k} \sum_{j=0}^{p} K_{1 j}  \tag{2.29}\\
& =(-1)^{k} \frac{-2}{\mu-1}
\end{aligned} \begin{aligned}
\left.\left(K D^{-1} \bar{B}^{T} D^{-1}\right)_{m k}\right|_{m \geq 2} & =\left.\sum_{j=0}^{p} K_{m j}\left(D^{-1} \bar{B}^{T} D^{-1}\right)_{j k}\right|_{m \geq 2} \\
& =\left.(-1)^{k}\left\{K_{m m}+K_{m m-2}\right\}\right|_{m \geq 2}=0 \tag{2.30}
\end{align*}
$$

The result for $K D^{-1} \bar{B}^{T} D^{-1}$ can be summarized:

$$
\left(K D^{-1} \bar{B}^{T} D^{-1}\right)_{m k}= \begin{cases}(-1)^{k} \frac{\mu-2 \phi_{1}^{p}}{\mu-1}, & m=0  \tag{2.31}\\ (-1)^{k} \frac{-2}{\mu-1}, & m=1 \\ 0, & \text { otherwise } .\end{cases}
$$

For $K D^{-1} \bar{C} D^{-1}$ it is obtained:

$$
\begin{align*}
&\left(K D^{-1} \bar{C} D^{-1}\right)_{0 k}=\sum_{j=0}^{p} K_{0 j}\left(D^{-1} \bar{C} D^{-1}\right)_{j k} \\
&= \begin{cases}2 \frac{\mu-(\mu+1) \phi_{1}^{p}}{2 \phi_{1}^{p},} & k \text { even },\end{cases}  \tag{2.32}\\
& \begin{aligned}
(K-1 & k \text { odd },
\end{aligned} \\
&== \begin{cases}-2 \frac{\mu+1}{\mu-1}, & k \text { even }, \\
2, & k \text { odd },\end{cases}  \tag{2.33}\\
& \begin{aligned}
\left(K D^{-1}\right)_{1 k} & =K_{1 j}\left(D^{-1} \bar{C} D^{-1}\right)_{j k}
\end{aligned}  \tag{2.34}\\
&=\left.2\left\{K_{m m}+K_{m m-2}\right\}\right|_{m \geq 2}=0
\end{align*}
$$

which can be summarized as:

$$
\left(K D^{-1} \bar{C} D^{-1}\right)_{m k}= \begin{cases}2 \frac{\mu-(\mu+1) \phi_{1}^{p}}{p^{p-1}}, & m=0, \quad k \text { even },  \tag{2.35}\\ 2 \phi_{1}^{p}, & m=0, \quad k \text { odd }, \\ -2 \frac{\mu+1}{\mu-1}, & m=1, \quad k \text { even }, \\ 2, & m=1, \quad k \text { odd }, \\ 0, & \text { otherwise. }\end{cases}
$$

Finally we evaluate $K D^{-1} I D^{-1}$ :

$$
\begin{align*}
\left(K D^{-1} I D^{-1}\right)_{0 k} & =\sum_{j=0}^{p} K_{0 j}\left(D^{-1} I D^{-1}\right)_{j k} \\
& =\sum_{j=0}^{p} K_{0 j} \frac{\delta_{j k}}{2 j+1}  \tag{2.36}\\
& =K_{0 k} \frac{1}{2 k+1} \\
& = \begin{cases}\frac{\mu}{\mu-1}-\frac{\mu+1}{\mu-1} \phi_{1}^{p}-\phi_{2}^{p}, & k=0, \\
\frac{1}{2 k+1}\left(\phi_{k}^{p}-\phi_{k+2}^{p}\right), & k \geq 1,\end{cases}
\end{align*}
$$

$$
\begin{align*}
\left(K D^{-1} I D^{-1}\right)_{1 k} & =\sum_{j=0}^{p} K_{1 j}\left(D^{-1} I D^{-1}\right)_{j k} \\
& =\sum_{j=0}^{p} K_{1 j} \frac{\delta_{j k}}{2 j+1} \\
& =K_{1 k} \frac{1}{2 k+1}  \tag{2.37}\\
& = \begin{cases}-\frac{\mu+1}{\mu-1}, & k=0, \\
1 / 3, & k=1, \\
0, & k \geq 2,\end{cases} \\
& =\begin{aligned}
\left.\left(K D^{-1} I D^{-1}\right)_{m k}\right|_{m \geq 2} & =\left.\sum_{j=0}^{p} K_{m j}\left(D^{-1} I D^{-1}\right)_{j k}\right|_{m \geq 2}
\end{aligned} \\
& =\left.\sum_{j=0}^{p} K_{m j} \frac{\delta_{j k}}{2 j+1}\right|_{m \geq 2}
\end{align*} \quad \begin{array}{ll}
\left.K_{m k} \frac{1}{2 k+1}\right|_{m \geq 2} \\
& =\left\{\begin{array}{ll}
-\frac{1}{2(m-1)-1}, & k=m-2, \quad m \geq 2, \\
2 m+1
\end{array},\right.  \tag{2.38}\\
0, & k=m, \quad m \geq 2,
\end{array},
$$

In summary:

$$
\left(K D^{-1} I D^{-1}\right)_{m k}= \begin{cases}\frac{K_{0 k}}{2 k+1} & m=0,  \tag{2.39}\\ -\frac{\mu+1}{\mu-1}, & m=1, \quad k=0 \\ 1 / 3, & m=1, \quad k=1 \\ -\frac{1}{2 m-3}, & m \geq 2, \quad k=m-2 \\ \frac{1}{2 m+1}, & m \geq 2, \quad k=m \\ 0, & \text { otherwise }\end{cases}
$$

At this stage we can calculate $K D^{-1}(\bar{K}-\lambda I) D^{-1}$ which is obtained by combining:

$$
\begin{align*}
K D^{-1}(\bar{K}-\lambda I) D^{-1} & =K D^{-1} \bar{A} D^{-1}-\mu^{-1}\left(K D^{-1} \bar{B} D^{-1}\right) \\
& -\epsilon\left[\mu^{-1}\left(K D^{-1} \bar{B} D^{-1}\right)-\left(K D^{-1} \bar{C} D^{-1}\right)+\mu\left(K D^{-1} \bar{B}^{T} D^{-1}\right)\right] \\
& -\lambda\left(K D^{-1} I D^{-1}\right) . \tag{2.40}
\end{align*}
$$

This can be written as:

$$
\begin{equation*}
K D^{-1}(-\bar{K}-\lambda I) D^{-1}=E+\epsilon E_{\epsilon}-\lambda\left(K D^{-1} I D^{-1}\right) \tag{2.41}
\end{equation*}
$$

where

$$
\begin{align*}
& E \equiv K D^{-1} \bar{A} D^{-1}-\mu^{-1}\left(K D^{-1} \bar{B} D^{-1}\right)  \tag{2.42}\\
& E_{\epsilon} \equiv \mu^{-1}\left(K D^{-1} \bar{B} D^{-1}\right)-\left(K D^{-1} \bar{C} D^{-1}\right)+\mu\left(K D^{-1} \bar{B}^{T} D^{-1}\right) \tag{2.43}
\end{align*}
$$

We have:

$$
\begin{align*}
& E_{0 k}= \frac{\mu}{\mu-1}-\frac{\mu+1}{\mu-1} \phi_{1}^{p}-\phi_{2}^{p}+\Phi_{k}^{p} \\
&-\mu^{-1} \frac{\mu}{\mu-1}\left(1-2 \phi_{1}^{p}\right) \\
&= 1-\phi_{1}^{p}-\phi_{2}^{p}+\sum_{m=1}^{2} \phi_{m}^{p}  \tag{2.44}\\
& \sum_{m=k+1}^{k+2}\left\{(-1)^{m-k}-1\right\} \phi_{m}^{p} \\
&= 1+\sum_{m=k+1}^{k+2}\left\{(-1)^{m-k}-1\right\} \phi_{m}^{p}=1-2 \phi_{k+1}^{p} .  \tag{2.45}\\
& E_{10}=-\frac{2 \mu}{\mu-1}+\mu^{-1} \frac{2 \mu}{\mu-1}=-2,  \tag{2.46}\\
&\left.E_{1 k}\right|_{k \geq 1}=-\frac{2}{\mu-1}+\mu^{-1} \frac{2 \mu}{\mu-1}=0,  \tag{2.47}\\
&\left.E_{m k}\right|_{m \geq 2}= \begin{cases}-2, & k=m-1, \\
0, & \text { otherwise. }\end{cases}
\end{align*}
$$

Note that in Eq.(2.44) $E_{0 p}=1$, because $\phi_{p+1}^{p}=0$ (Eq.(2.3)). In summary, we obtain for $E$ :

$$
E_{m k}= \begin{cases}1-2 \phi_{k+1}^{p} & m=0,  \tag{2.48}\\ -2, & m \geq 2, \quad k=m-1 \\ 0, & \text { otherwise }\end{cases}
$$

Next we evaluate $E_{\epsilon}$ :

$$
\begin{align*}
\left.\left(E_{\epsilon}\right)_{0 k}\right|_{k \text { even }}= & \mu^{-1} \frac{\mu}{\mu-1}\left(1-2 \phi_{1}^{p}\right)-2 \frac{\mu-(\mu+1) \phi_{1}^{p}}{\mu-1}  \tag{2.49}\\
& +\mu \frac{\mu-2 \phi_{1}^{p}}{\mu-1}=\mu-1 \\
\left.\left(E_{\epsilon}\right)_{0 k}\right|_{k \text { odd }}= & \mu^{-1} \frac{\mu}{\mu-1}\left(1-2 \phi_{1}^{p}\right)-\phi_{1}^{p}  \tag{2.50}\\
& -\mu \frac{\mu-2 \phi_{1}^{p}}{\mu-1}=-(\mu+1)  \tag{2.51}\\
\left.\left(E_{\epsilon}\right)_{1 k}\right|_{k \text { even }}= & -\mu^{-1} \frac{2 \mu}{\mu-1}+2 \frac{\mu+1}{\mu-1}-\mu \frac{2}{\mu-1}=0  \tag{2.52}\\
\left.\left(E_{\epsilon}\right)_{1 k}\right|_{k \text { odd }}= & -\mu^{-1} \frac{2 \mu}{\mu-1}-2+\mu \frac{2}{\mu-1}=0  \tag{2.53}\\
\left.\left(E_{\epsilon}\right)_{m k}\right|_{m \geq 2}= & 0
\end{align*}
$$

In summary, we obtain for $E_{\epsilon}$ :

$$
\left(E_{\epsilon}\right)_{m k}= \begin{cases}(-1)^{k} \mu-1, & m=0  \tag{2.54}\\ 0, & \text { otherwise }\end{cases}
$$

From Eqs.(2.39), (2.54) and (2.48) we can evaluate $E-\epsilon E_{\epsilon}-\lambda K D^{-1} I D^{-1}=K D^{-1}(\bar{K}-$ $\lambda I) D^{-1}$ :

$$
\begin{align*}
\left(K D^{-1}(\bar{K}-\lambda I) D^{-1}\right)_{00} & =1-2 \phi_{1}^{p}-\epsilon(\mu-1)-\lambda K_{00} \\
& =1-2 \phi_{1}^{p}-\epsilon(\mu-1)-\lambda\left[\frac{\mu}{\mu-1}-\frac{\mu+1}{\mu-1} \phi_{1}^{p}-\phi_{2}^{p}\right] . \tag{2.55}
\end{align*}
$$

From Eq.(2.3) we evaluate for $k=0$ :

$$
\begin{equation*}
\left[1-2 \phi_{1}^{p}-\epsilon(\mu-1)\right]=\lambda\left(\phi_{0}^{p}-\phi_{2}^{p}\right) \tag{2.56}
\end{equation*}
$$

Upon substitution of Eq.(2.56) into Eq.(2.55) we obtain:

$$
\begin{equation*}
\left(K D^{-1}(\bar{K}-\lambda I) D^{-1}\right)_{00}=\frac{\lambda}{\mu-1}\left[(\mu-1) \phi_{0}^{p}+(\mu+1) \phi_{1}-\mu\right] \tag{2.57}
\end{equation*}
$$

Additionally:

$$
\begin{align*}
\left.\left(K D^{-1}(\bar{K}-\lambda I) D^{-1}\right)_{0 k}\right|_{k \geq 1} & =1-2 \phi_{1}^{p}-\epsilon\left[(-1)^{k} \mu-1\right]-\frac{\lambda}{2 k+1} K_{0 k} \\
& =1-2 \phi_{1}^{p}-\epsilon\left[(-1)^{k} \mu-1\right]-\frac{\lambda}{2 k+1}\left[\phi_{k}^{p}-\phi_{k+2}^{p}\right] . \tag{2.58}
\end{align*}
$$

Employing Eq.(2.3) we have:

$$
\begin{equation*}
\frac{\lambda}{2 k+1}\left[\phi_{k}^{p}-\phi_{k+2}^{p}\right]=1-2 \phi_{1}^{p}-\epsilon\left[(-1)^{k} \mu-1\right], \tag{2.59}
\end{equation*}
$$

and clearly:

$$
\begin{equation*}
\left.\left(K D^{-1}(\bar{K}-\lambda I) D^{-1}\right)_{0 k}\right|_{k \geq 1}=0 . \tag{2.60}
\end{equation*}
$$

Furthermore

$$
\left.\left(K D^{-1}(\bar{K}-\lambda I) D^{-1}\right)_{m k}\right|_{m \geq 1}= \begin{cases}-2+\lambda \frac{\mu+1}{\mu-1}, & m=1, \quad k=0,  \tag{2.61}\\ -\frac{\lambda}{3}, & m=1, \quad k=1, \\ \frac{\lambda}{2 m-3}, & m \geq 2, \quad k=m-2, \\ -2, & m \geq 2, \quad k=m-1, \\ -\frac{\lambda}{2 m+1}, & m \geq 2, \quad k=m \\ 0, & \text { otherwise } .\end{cases}
$$

From Eqs.(2.57), (2.60) and (2.61) it is observed that $K D^{-1}(\bar{K}-\lambda I) D^{-1}$ is a lower triangular matrix:

$$
\begin{equation*}
\left(K D^{-1}(\bar{K}-\lambda I) D^{-1}\right)_{m k}=0, \quad k>m \tag{2.62}
\end{equation*}
$$

As a result, the determinant of $K D^{-1}(\bar{K}-\lambda I$ is equal to the product of its diagonal entries, which upon using Eq.(2.57) and Eq.(2.61), yields:

$$
\begin{equation*}
\operatorname{det}\left(K D^{-1}(\bar{K}-\lambda I) D^{-1}\right)=\frac{\lambda}{\mu-1}\left[(\mu-1) \phi_{0}^{p}+(\mu+1) \phi_{1}-\mu\right] \prod_{m=1}^{p} \frac{-\lambda}{2 m+1} \tag{2.63}
\end{equation*}
$$

Hence the characteristic polynomial we are looking for is given by Eq.(2.2):

$$
\left[(\mu-1) \phi_{0}^{p}+(\mu+1) \phi_{1}-\mu\right] \lambda^{p+1}=0
$$

which completes the proof.

# INTEGRATION IN THE COMPLEX PLANE 

In this appendix mathematical techniques are presented, which are used in chapter 6 to evaluate Fourier integrals in the complex plane. Only a short description of the techniques will be given. For a thorough description and formal proofs the reader is referred to the literature. In each section the references, from which the text is derived, are included.

## C. 1 Calculus of residues

See also, amongst others, references: [26], [27], [61], [64] and [71].
For $z$ and $f(z) \in \mathbb{C}$, assume that $f(z)$ is analytic ${ }^{1}$ on and inside a closed contour $C$, except at the isolated points $a_{1}, a_{2}, \ldots, a_{n}, n \in \mathbb{N}<\infty$. Then

$$
\begin{equation*}
\frac{1}{2 \pi i} \oint_{C} f(z) d z=\sum_{k=1}^{n} \operatorname{Res}_{z=a_{k}} f(z) . \tag{3.1}
\end{equation*}
$$

## C.1.1 Calculation of residues

1 Employing Laurent series expansion, see a.o. [27], [64]. If $f(z)$ is analytic in the annulus $R_{1}<\left|z-z_{0}\right|<R_{2}$, then it can be expressed as:

$$
\begin{equation*}
f(z)=\ldots+\frac{b_{3}}{\left(z-z_{0}\right)^{3}}+\frac{b_{2}}{\left(z-z_{0}\right)^{2}}+\frac{b_{1}}{\left(z-z_{0}\right)}+a_{0}+a_{1}\left(z-z_{0}\right)+a_{2}\left(z-z_{0}\right)^{2}+\ldots \tag{3.2}
\end{equation*}
$$

which converges for $R_{1}<\left|z-z_{0}\right|<R_{2}$. the Laurant series expansion implies that a function which is analytic in the annulus $R_{1}<\left|z-z_{0}\right|<R_{2}$ can be expressed as a sum of two functions, the one involving positive powers being analytic inside the circle $\left|z-z_{0}\right|<R_{2}$, and the other analytic outside the circle $\left|z-z_{0}\right|>R_{1}$ ([27]).

2 Simple pole:

$$
\begin{equation*}
\operatorname{Res}_{z=a} f(z)=\lim _{z \rightarrow a}(z-a) f(z) \text {. } \tag{3.3}
\end{equation*}
$$

[^5]More general, if $f(z)$ and $g(z)$ analytic and $f(a) \neq 0, g(a)=0$ and $g^{\prime}(a) \neq 0$, then:

$$
\begin{equation*}
\operatorname{Res}_{z=a} \frac{f(z)}{g(z)}=\frac{f(a)}{g^{\prime}(a)} . \tag{3.4}
\end{equation*}
$$

3 Pole of order $m$ :

$$
\begin{equation*}
\operatorname{Res}_{z=a} f(z)=\lim _{z \rightarrow a} \frac{1}{(m-1)!}\left(\frac{d}{d z}\right)^{m-1}(z-a)^{m} f(z) \tag{3.5}
\end{equation*}
$$

## C.1.2 Infi nitely many poles

For this section Papoulis [61] was used as reference. In [61] this subject is discussed in relation to the inverse Laplace transform. Here we will present the description for the inverse Fourier transform. The evaluation of the inverse Fourier transform of $\tilde{f}(\omega)$ is performed in the complex plane. For $C$ a semi-circle in the complex plane with radius $R$, such as the contours presented in Fig.(3.1):

$$
\begin{equation*}
f(t)=\frac{1}{2 \pi} \oint_{\Gamma} \tilde{f}(\omega) e^{i \omega t} d \omega \tag{3.6}
\end{equation*}
$$

where $\Gamma=C \cup[-R, R]$ encloses all poles.
The case when the integrand has infinitely many poles, as in Fig.(3.1), which are not confined to a finite region of the complex plane, requires special consideration. Let us assume that a sequence of circular arcs $C_{1}, C_{2}, \ldots, C_{n}, \ldots$, with radii tending to infinity can be found, see also Fig.(3.1), such that:

$$
\begin{equation*}
\tilde{f}(\omega) \rightarrow 0 \quad \text { as } \quad \omega \rightarrow \infty \quad \omega \in \mathbb{C}_{n} \tag{3.7}
\end{equation*}
$$

Applying Jordan' Lemma, appendix C.3, to the integrals along these $\operatorname{arcs} C_{n}$, we have:


FIGURE 3.1: Sequence of circular arcs enclosing infi nite poles.

$$
\begin{equation*}
\oint_{\Gamma_{n}} \tilde{f}(\omega) e^{i \omega t} d \omega \underset{n \rightarrow \infty}{\longrightarrow} \quad t>0 . \tag{3.8}
\end{equation*}
$$

We obtain:

$$
\begin{equation*}
f(t)=\frac{1}{2 \pi} \lim _{n \rightarrow \infty} \oint_{\Gamma_{n}} \tilde{f}(\omega) e^{i \omega t} d \omega \quad t>0 . \tag{3.9}
\end{equation*}
$$

The integral in the last equation can be evaluated again by applying the residue theorem.

## C. 2 Causality condition

For a more elaborate description of the causality condition the reader is referred to Rienstra \& Hirschberg [70]. Other references which might be helpful are by Crighton et al. [27] and Papoulis [61]. In problems where the (sound) field is induced by a forcing term that is zero for $t<t_{0}$ and starts at $t=t_{0}$, it is argued on physical grounds that the response cannot anticipate the source ([27]). Therefore we impose the causality condition; for example for the pressure (perturbation):

$$
\begin{equation*}
p=0 \text { for } t<t_{0} . \tag{3.10}
\end{equation*}
$$

Often it is perfectly clear how the causality condition has to be imposed. However, when the solution involves Fourier transformations it is not clear anymore. For the Fourier transformation of $p(t)$, i.e. $\tilde{p}(\omega)$, we have the following theorem ([70], [61]):

## Theorem C. 1 (Causality condition) If:

- $\tilde{p}(\omega)$ analytic in $\operatorname{Im}(\omega) \leq 0$,
- $\left|\tilde{p}(\omega)^{2}\right|$ integrable,
- $\exists t_{0} \in \mathbb{R}$, such that $e^{i \omega t_{0}} \tilde{p}(\omega) \rightarrow 0$ for $\omega \rightarrow-i \infty$,


## Then:

- $p(t)$ is causal and vanishes for $t<t_{0}$.

The condition that $\left|\tilde{p}(\omega)^{2}\right|$ has be integrable, corresponds to $p(t)$ having finite energy [61].
Causal functions If a function $p$ is zero for $t<t_{0}$, then the real and imaginary parts (or amplitude and phase) of its Fourier integral $\tilde{p}(\omega)$ are not independent of each other [61]. In fact, if one of these quantities, $\operatorname{Re}(\tilde{p}(\omega))$ or $\operatorname{Im}(\tilde{p}(\omega))$, is specified in a certain part of the $\omega$ axis and the other in the remaining part of this axis, then $\tilde{p}(\omega)$ can be uniquely found.

## C. 3 Lemma of Jordan

The Lemma of Jordan is presented by, amongst others, Crighton et al. [27] and Papoulis [61]. It deals with integrands of the form $e^{i t z} f(z)$ for $t>0$, which are common in Fourier transforms, evaluated around semi-circles in the upper half plane. The Lemma exploits the fact that $\left|e^{i t z}\right|=e^{-t \operatorname{Im}(z)}$ is exponentially small on such a contour.

Let $t>0$ and

$$
\begin{equation*}
f(z) \rightarrow 0, \quad \text { with } z \rightarrow \infty \tag{3.11}
\end{equation*}
$$

then:

$$
\begin{equation*}
\int_{S} e^{i t z} f(z) d z \rightarrow 0, \quad \text { with } r \rightarrow \infty \tag{3.12}
\end{equation*}
$$

where $S$ is the semi-circle in the upper half plane.

## C. 4 Method of Stationary Phase

The method of Stationary Phase is described, amongst others, by Crighton et al. [27] and Papoulis [61]. A thorough description can also be found in Bleistein \& Handelsman [13].

The method of Stationary Phase can be used to obtain asymptotic expansions as $\lambda \rightarrow \infty$ of integrals of the form:

$$
\begin{equation*}
I(\lambda)=\int_{-\infty}^{\infty} g(\omega) e^{i \lambda \phi(\omega)} d \omega \tag{3.13}
\end{equation*}
$$

It is noted that the method is also applicable when the interval of integration is finite, say $\omega \in[a, b]$, but more about that can be found in the literature.

The next part is taken from Bleistein \& Handelsman [13]. Suppose that $\omega=c$ is a point at which the derivative of $\phi, \phi^{\prime}(c)$, does not vanish. Then there exists a small neighborhood $N_{c}$ of this point such that, as $\omega$ varies throughout $N_{c}, \phi(\omega)$ is changing. If furthermore $\lambda$ is large, then the change in $\lambda \phi$ is rapid, so that the oscillations of the real and imaginary parts of $e^{i \lambda \phi}$ about zero are rapid. Now consider:

$$
\begin{equation*}
I_{c}(\lambda)=\int_{N_{c}} g(\omega) e^{i \lambda \phi(\omega)} d \omega . \tag{3.14}
\end{equation*}
$$

Because $N_{c}$ is a small interval about $\omega=c$, we may closely approximate $g(\omega)$ by $g(c)$. Then, upon assuming $\lambda$ large, we find that the rapid oscillations of $e^{i \lambda \phi}$ produce cancellations which, in turn, tend to decrease the value of $I_{c}(\lambda)$.

Assume now that $\phi^{\prime}$ does vanish at $\omega=c$. Then no matter how large $\lambda$, there exists a neighborhood $N_{c}(\lambda)$ of $\omega=c$ throughout which $\lambda \phi$ does not change rapidly. As $\omega$ varies in $N_{c}(\lambda), e^{i \lambda \phi}$ does not oscillate rapidly and cancellation does not occur. It is therefore anticipated that the result of the integral Eq.(3.14), for $\lambda$ large, depends primarily on the behavior of $g$ and $\phi$ near points at which $\phi^{\prime}=0$. In calculus these points are known as stationary points of $\phi$. Assume that $\phi^{\prime \prime} \neq 0$, then the integral of Eq.(3.14) can be approximated by:

$$
\begin{equation*}
I(\lambda) \approx g(c) \sqrt{\frac{2 \pi}{\lambda\left|\phi^{\prime \prime}(c)\right|}} e^{i \lambda \phi(c)} e^{\frac{i \pi}{4} \operatorname{sgn}\left(\phi^{\prime \prime}(c)\right)} . \tag{3.15}
\end{equation*}
$$

## SUMMARY

This thesis concerns the development of a numerical method for the accurate simulation of the propagation of acoustic information through fluids moving with non-uniform velocity in three-dimensional complex domains.

The development of the numerical method involves the identification of the equations governing the propagation of sound, the identification of a suitable computational algorithm for solving the governing equations and verification of the numerical method.

Before introduction of the numerical method, it has been shown, in chapter 2, that the propagation of (aero-) acoustic information through non-homogeneous moving fluids can be described by the linearized Euler equations, assuming that there is no feedback from the acoustic field to the background flow. In chapter 2 the linearized Euler equations have been derived from the more general Navier-Stokes equations. In literature a variety of different sets of equations is in use which are all designated as the linearized Euler equations. The observed differences in literature may be caused by different or additional assumptions which have been made. Another cause may be that the Euler equations can be written in different forms before introducing the linearization process. In chapter 2 it has been shown that a linear set of equations can be derived from the Euler equations in various different ways. It is shown that, by rewriting one of these sets of equations, one of the other formulations can be obtained without any additional assumptions and/or approximations.

Inherently to using numerical methods, numerical dissipation and dispersion are introduced. Numerical dissipation unphysically dampens the amplitude of the propagating wave and numerical dispersion alters the propagation velocity in an unphysical manner. Numerical methods applied for the simulation of the propagation of acoustic information therefore require special attention. After considering several computational methods the discontinuous Galerkin finite element method was found best suited for achieving our objectives. The discontinuous Galerkin method, which is an extremely compact finite element method, can be used to obtain higher-order formal accuracy, which minimizes numerical dissipation and dispersion. Because of its compactness, the discontinuous Galerkin method can be applied to domains involving complex geometries. The accuracy of the method does not depend on the smoothness of the mesh. The accuracy in an element only depends on the size and shape of the element and the degree to which the solution is approximated (Atkins \& Shu [6]). Furthermore, the method is well suited for implementation on parallel computer platforms and because inter-element continuity is not required, the discontinuous Galerkin method is a good candidate for adaptive error control, such as local grid refinement (h-refinement) and local polynomial-degree-variation (p-refinement). In addition, when applying the discontinuous

Galerkin method the treatment of boundary conditions is simple. In this thesis the linearized Euler equations for the primitive perturbation variables are numerically solved on tetrahedral meshes employing the Discontinuous Galerkin method for the spatial discretization and the multi-stage low-storage Runge-Kutta method for the time discretization. Coupling between elements is provided by the Lax-Friedrichs flux. In chapter 3 a detailed description of the discontinuous Galerkin method has been presented.

First step in the verification of the computational method has been the wave-propagation and stability analysis which has been presented in chapter 4 . The wave propagation properties have been identified for the semi-discrete system which is obtained from applying the discontinuous Galerkin method on a uniform mesh for the spatial discretization of a onedimensional model equation, i.e. the scalar advection equation. The analysis has been conducted by comparing the eigenvalues of the numerical dispersion relation, a functional relation between the angular frequency of a wave and the wave numbers of the spatial variable, of the semi-discrete system with the exact dispersion relation of the scalar advection equation.

In literature the wave propagation analysis has been limited to a certain value of $p$, where $p$ is the highest degree of the basis functions which are used in the discontinuous Galerkin formulation. In addition, the analysis is often restricted to analyzing the semi-discrete system which is obtained by using the exact solution of the Riemann problem at element interfaces. In chapter 4 the characteristic polynomial is identified for any value of $p$ and for both the case in which the exact solution of the Riemann problem is used and the case in which the Lax-Friedrichs flux is used as approximate Riemann solver. The $p+1$ eigenvalues are obtained by numerically solving the characteristic polynomial, which is a complex polynomial of degree $p+1$. It is shown that there is always one eigenvalue which approximates the exact dispersion relation best. This eigenvalue is called the physical mode. The other eigenvalues (or modes) represent waves which experience more dissipation and dispersion than the physical mode. All waves represented by the eigenvalues experience increasing amounts of dissipation and dispersion with increasing wave number. For the maximum wave number, which is given by the minimum resolvable wave length on the considered mesh, all modes, including the physical one, experience the most dispersion and dissiption. The least accurate mode, which has large damping, propagates in the wrong direction for a large range of wave numbers. It is furthermore shown that the dissipation and dispersion errors, obtained for the physical mode, converge at a rate of $h^{2 p+2}$ and $h^{2 p+3}$, where $h$ is the uniform element mesh size. Hu \& Atkins [42] report convergence rates of $h^{2 p+1}$ and $h^{2 p+2}$ for the dissipation and dispersion errors, respectively. Because of the high convergence rates of the dissipation and dispersion errors, the discontinuous Galerkin method is very well suited for application to acoustic propagation problems.

The last three chapters of this thesis, chapters 5 to 7 , report on the numerical verification of the developed algorithm.

In chapter 5 numerical results obtained for the convection of a two-dimensional Gaussian pulse in a uniform mean flow, are compared with the obtained analytical solution to the problem. The considerd problem is only slightly different from a problem described in the ICASE/LaRC Workshop on Benchmark Problems in Computational Aeroacoustics ([3]). The results have been obtained while using polynomial basis functions in the discontinuous Galerkin method of degree $\leq 1$. In the low-storage Runge-Kutta time integration method four stages have been used, with the coefficients chosen such that the time integration is fourth order accurate in terms of the truncation error. The obtained results are shown to be in good
agreement with the analytical solution.
It is furthermore shown that the obtained numerical solution converges at a rate at least $h^{2 \frac{1}{2}}$, measured in a norm defined on all common node points of different considered meshes (ranging from coarse to fine), where $h$ is a characteristic mesh size. In addition, a performance test of the parallelized algorithm implemented on the super computer TERAS, showed that (near) linear speed-up is obtained for up to 32 processors on a mesh of approximately 0.6 million tetrahedral elements.

In chapter 6 the analytical solution to the problem of acoustic radiation from a vibrating wall segment inside an infinite rectangular duct has been presented. The amplitude of the wall vibration is assumed small compared to the acoustic wave length and the surface dimensions and it is assumed that the problem can be described by the linearized Euler equations. The mathematical solution procedure used to obtain the analytical solution is based on the mathematical procedure presented by Kuijpers et al. [50]. The mathematical model described by Kuijpers et al. [50] has been developed for use in the design process of MRI scanners. In developing the mathematical procedure the MRI scanner was modelled by a cilindrical duct of finite length. In many well-known textbooks, e.g. Crighton [27], Jones [46], Morse \& Ingard [59], Morse \& Feshbach [60] and Pierce [63], the subject of propagation and diffraction of sound waves in ducts (usually cilindrical ducts) is covered. In the textbook by Pierce [63] one chapter is devoted to the problem of radiation from vibrating bodies and another to the radiation from sources near and on solid surfaces. Filippi et al. [32] and Morse \& Feshbach [60] describe the problem of a vibrating piston, located at one of the ends of the duct, which generates sound in the duct. Also radiation from (point) sources in ducts is described in literature, however, the vibration of a finite part of the wall that radiates sound into the rectangular duct has, to the author's knowledge, not been covered as such. The mentioned textbooks have, however, been of great importance for the understanding of the problem.

In chapter 6 the analytical solution is obtained for two types of background flows, viz. no flow and a background flow with uniform mean flow velocity in axial direction. The analytical solution, which is presented in the time domain, encompasses one propagating mode, of which the analytical solution can be obtained in closed form, and an infinite number of diffracted waves. For the diffracted waves only an approximate solution has been presented, which has been obtained employing the method of stationary phase.

In chapter 7 numerical results are presented obtained from numerically solving the problem of acoustic radiation from a vibrating wall segment inside an infinite rectangular duct. The numerical solutions have been compared with the analytical solutions and have been shown to be in good agreement.

It has been shown that the amplitude of the plane wave solution is (much) larger than the amplitudes of the diffracted waves. The amplitude of the diffracted waves have been shown to decrease when moving further and further away from the vibrating wall.

SAMENVATTING

## ACKNOWLEDGMENT

ABOUT THE AUTHOR


[^0]:    ${ }^{1}$ A more detailed list of acoustical topics is provided by the American Institute of Physics at the internet site "www.aip.org/pubservs/pacs.html" or can be found in each volume of the Journal of the Acoustical Society of America.

[^1]:    ${ }^{1}$ Should we have chosen $\gamma R$ as a scaling quantity, instead of $c_{v}$, we would have obtained here the relation: $\lambda f=c_{0}$.

[^2]:    ${ }^{1}$ Cockburn et al. present a detailed historical overview of the Discontinuous Galerkin method [19]. The overview not only encompasses DG methods for solving linear and scalar equations, but also for solving non-linear systems, such as the Euler and Navier-Stokes equations. The reader is referred to [19] for a more elaborate overview and description of the different methods based on the DG formulation.

[^3]:    ${ }^{1}$ The parallel version of the computational method has been developed in close cooperation with Dr. E. Védy of TNO-TPD.

[^4]:    ${ }^{2}$ A Banach space is a complete normed space. A normed space is a vector space with a norm defi ned on it [47].

[^5]:    ${ }^{1}$ Defi nition: The function $f(z)$ is analytic on a domain $\Omega$ if $f(z)$ is differentiable at every point of $\Omega$.

