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Multi-Level Wave-Ray solution of 2D-Helmholtz equation

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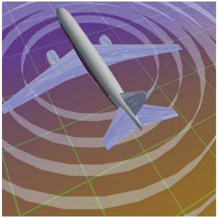
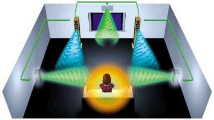
Acoustics

- Structural acoustics
- Aero acoustics
  - Interior acoustics
  - Exterior acoustics
- Mixed
- Noise and sound control important
  - Noise: Aircraft, traffic, cruise ship
  - Sound: Cinema, theatre, home sound systems





Acoustics

- Conservation of mass, momentum and energy
- Viscous and heat conducting effects neglected
- Adiabatic process
  - 140 [dB] fluctuations of 200 [Pa]
  - Atmospheric pressure 10^5 [Pa]
- Small perturbations $\longrightarrow$ linear acoustics

Helmholtz equation

- Linear acoustics $\longrightarrow$ Wave-equation
- Separation of variables $\longrightarrow$ Standing wave solutions
  $p(\mathbf{x}, t) = u(\mathbf{x})\, g(t)$
- Superposition $\longrightarrow$ One equation per frequency ω
- Helmholtz equation for $u(\mathbf{x})$:
  - $\nabla^2 u(\mathbf{x}) + k(\mathbf{x})^2 u(\mathbf{x}) = f(\mathbf{x})$
  - Wave-number k for frequency and speed of sound:
    $k^2 = \frac{\omega^2}{c_0(\mathbf{x})^2}$
  - Sources of sound $f(\mathbf{x})$ for frequency

- Harmonic function
  k periods per 2π
- Numerical solution required
- Finite Difference Method:
  $\frac{u_{i-1}^h - 2u_i^h + u_{i+1}^h}{h} + k_i^h u_i^h = f_i^h$
- System of algebraic equations:
  $\mathbf{A} \cdot \mathbf{u}^h = \mathbf{f}^h$
    - Direct method: exact
    - Iterative method: approximation
- Large kL requires fine mesh

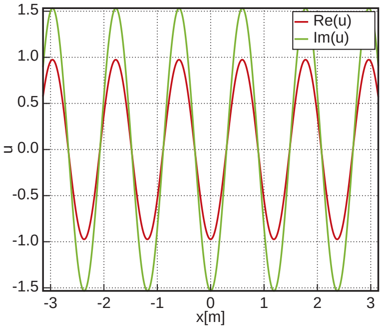


Figure: Solution $k = 5.3$, $L = 2\pi$

Helmholtz equation

- Iterative techniques are inefficient
- Multi-Level techniques for efficient solution
- Standard Multi-Level schemes fail for Helmholtz
- Wave-Ray scheme improves the efficiency
- Separation of rays required for Wave-Ray scheme

Objective

Develop Wave-Ray algorithm for 2D-Helmholtz equation:

- Extend existing 1D algorithm to non-homogeneous case
- Produce scheme for ray separation in 2D space
- Build 2D Wave-Ray algorithm

Iterative method

- Start with proper initial approximation $\tilde{\mathbf{u}}^h$
- Correct approximation to reach new approximation $\hat{\mathbf{u}}^h$
- Residual is difference in equations: $\mathbf{r} = \mathbf{f}^h - \mathbf{A} \cdot \hat{\mathbf{u}}^h$
- Amplification of residual per sweep in approximation usually $1 - \mathcal{O}\left(h^s\right)$
- For Helmholtz error amplification depends on h and k

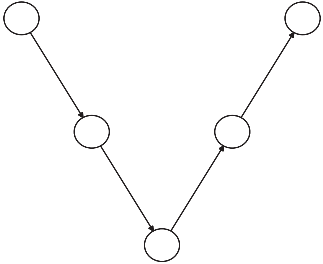
Iterative method

Two methods used:

- Gauß-Seidel: Stable on coarse grids
- Kaczmarz:
  - Solves projection of $\mathbf{u}^h$:
    $\left(\mathbf{A} \cdot \mathbf{A}^\mathsf{T}\right) \cdot \mathbf{y}^h = \mathbf{f}^h$, with $\mathbf{A}^\mathsf{T} \cdot \mathbf{y}^h = \mathbf{u}^h$
  - Always stable but slower

Multi-Level method

- Residual reduction per sweep $1 - \mathcal{O}\left(h^s\right) \rightarrow$ coarse grid residual reduction better
- Fine grids cannot reduce low frequen errors efficient
- Coarse grids are used to reduce these error components



Residual for standard iterative method after each sweep for $k = 0$
Residual for Multi-Level method after each cycle for $k = 0$

Inter-grid operators

- Interpolation from coarse grid to fine grid, index $I = 2i$:

$$I_H^h \langle\rangle = \frac{1}{2} \left[\begin{array}{ccc} 1 & 2 & 1 \end{array}\right] \rightarrow$$

$$I_H^h \left\langle v_I^H \right\rangle \Rightarrow v_{i-1}^h = \frac{v_{I-1}^H + v_{I+1}^H}{2},\ v_i^h = v_I^H,\ v_{i+1}^h = \frac{v_I^H + v_{I+1}^H}{2}$$

- Restriction fine to coarse grid by full weighting:

$$I_h^H \langle\rangle = \frac{1}{4} \left[\begin{array}{ccc} 1 & 2 & 1 \end{array}\right] \rightarrow I_h^H \left\langle u_i^h \right\rangle = \frac{1}{4}\left(u_{i-1}^h + 2u_i^h + u_{i+1}^h\right)$$

Full Approximation Scheme(FAS)

- Equation coarse grid for Full Approximation Scheme:

$$I_h^H \langle \mathbf{A} \rangle \cdot \mathbf{u}^H = I_h^H \left\langle \mathbf{f}^h - \mathbf{A} \cdot \tilde{\mathbf{u}}^h \right\rangle + I_h^H \langle \mathbf{A} \rangle \cdot I_h^H \left\langle \tilde{\mathbf{u}}^h \right\rangle$$

- Correction of fine grid solution:

$$\hat{\mathbf{u}}^h = \tilde{\mathbf{u}}^h + I_H^h \left\langle \mathbf{u}^H - I_h^H \left\langle \tilde{\mathbf{u}}^h \right\rangle \right\rangle$$

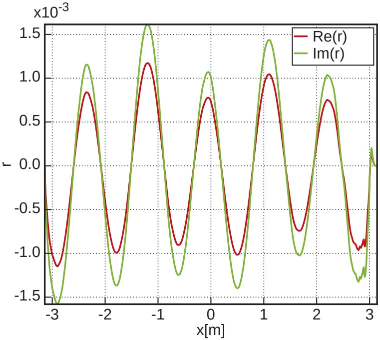


Figure: Residual $k = 5.3$, $L = 2\pi$

Wave-Ray Principle

- Frequency solution remains in residual
- Different cycle for reducing these components
- Introduction of ray equations for this process
- Separation scheme for rays required

Ray equations

- Ray equations for amplitude harmonic functions in solution
  $$u = a(x)\, e^{\iota s(x)} + b(x)\, e^{-\iota s(x)} \text{ with } s(x) = \int_{x_a}^{x_b} k(x)\ dx$$
- 1D:
  - Substitution in Helmholtz leads to:
    $$\frac{d^2 a}{dx^2} + \iota\left(\frac{d(ak)}{dx} + k\frac{da}{dx}\right) = f_a,\ \frac{d^2 b}{dx^2} - \iota\left(\frac{d(bk)}{dx} + k\frac{db}{dx}\right) = f_b$$
    with $f_a e^{\iota s(x)} + f_b e^{-\iota s(x)} = f$
  - f_a and f_b unknown for arbitrary forcing
  - Boundary conditions represent sound radiated into domain

Ray equations

- 2D:
  - Ray equations for all directions
  - Representation with eight directions sufficient on grids with $kh \approx 1$
- Ray solution interpolated grid with $kh = 4$ to $kh = 1$ via grid with $kh = 2$
- Ray solution corrects Helmholtz solution on grid with $kh \approx 1$
  $\hat{u}_i^h = \tilde{u}_i^h + \left(\hat{a}_i^h - \tilde{a}_i^h\right) e^{\iota s_i} + \left(\hat{b}_i^h - \tilde{b}_i^h\right) e^{-\iota s_i}$
- Separation of components in residual provides for right hand sides

Wave-Ray scheme

- Start with standard Multi-Level cycle; Wave Cycle
  - Coarsest grid $kh \approx 4$
  - No relaxation on grids with $1 \leq kh \leq 2.8$
- Subsequently Ray Cycle
  - Restriction of functions to $kh \approx 1$
  - Separation process to ray grid with $kh \approx 4$
  - Boundary conditions introduced with ray equations
  - Interpolation ray, correction wave on $kh \approx 1$

Separation for 1D

- Residual harmonic function: $r_i = r_{a_i} e^{\iota s_i} + r_{b_i} e^{-\iota s_i}$
- Multiplication with inverse ray component:
  $r_i e^{-\iota s_i} = r_{a_i} + r_{b_i} e^{-2\iota s_i}$
- Frequency relative to mesh $2k_i h$
- Full weighting to grid with $kh \approx 2$ such that:
  - Constant components remain constant
  - Components with $2k_i h$ eliminated
- Injection of result to grid with $kh \approx 4$

Example for 1D

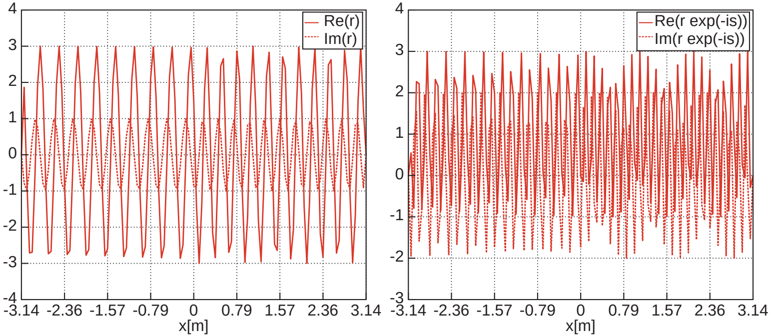


Figure: Test residual $r = e^{\iota s(x)} + 2e^{-\iota s(x)}$ obtains twice frequency after multiplication

Example for 1D

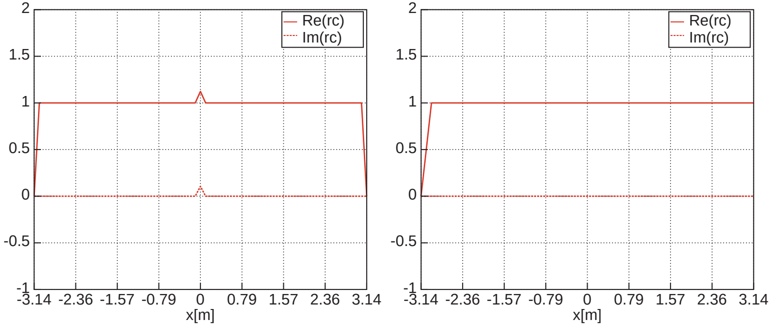
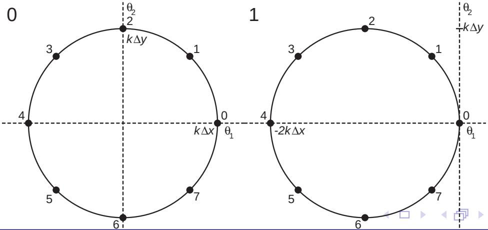


Figure: Amplitude almost exact after weighing, exact after injection to coarser grid

Separation for 2D

- Process similar to 1D, Circle of frequencies
- Repeated weighing to eliminate specific directions
- Domain extended in last step to avoid boundary influences

Figure: Frequencies in all directions, shifted after multiplication



Example for 2D

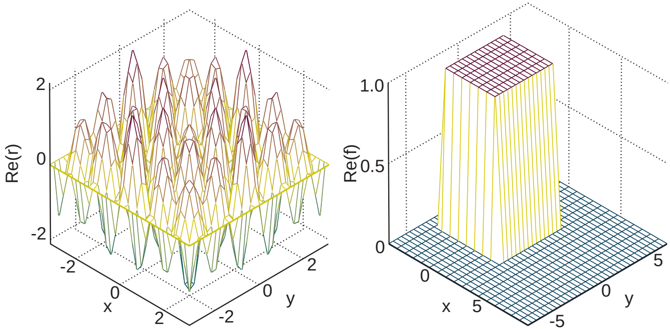


Figure: Separation for test residual $r = \sum_{n=0}^{7} e^{\iota k \xi_n}$ returns exact amplitude

Result for 1D

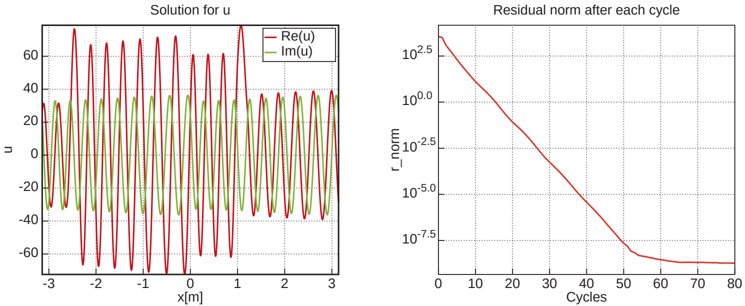


Figure: L_2-norm of residual reduces fast for case with sources and varying wave-number

Result for 2D

- Only four rays implemented
- Diagonal rays cause process to stall
- Fast reduction of residual in implemented directions

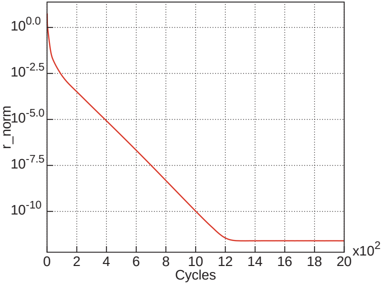


Figure: Residual norm for 2D case

Conclusions and recommendations

- Conclusions:
  - Separation for 1D and 2D is possible
  - 1D Wave-Ray scheme shows good performance
  - 2D scheme works for $k = 2.6$ with four rays
  - Initial performance 2D promising
- Recommendations:
  - 2D scheme needs extension to eight rays

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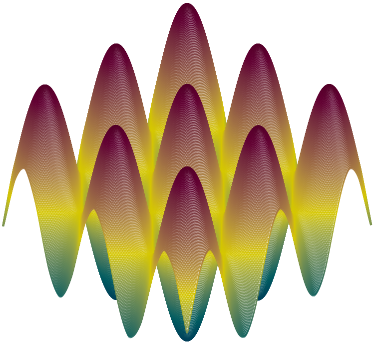
  - 2D scheme works for $k = 2.6$ with four rays
  - Initial performance 2D promising
- Recommendations:
  - 2D scheme needs extension to eight rays
  - Parameter study for best setup 2D
  - Extension 3D for practical use and experimental validation



Programma

- 14.00-±15.45 Afstudeer colloquium
- ±15.45-±16.30 Borrel in Diepzat tijdens besloten ondervraging
- ±16.30-±17.00 Diploma-uitreiking? in Z203
- ±17.00-17.55 Verder borrelen in diepzat
- 18.00-… Borrelen en lichte maaltijd op Matenweg 32