Kinematic and static assumptions for homogenization in micromechanics of granular materials

N.P. Kruyt
Department of Mechanical Engineering
University of Twente, The Netherlands
n.p.kruyt@wb.utwente.nl

L. Rothenburg
Department of Civil Engineering
University of Waterloo, Canada

Overview

- Micromechanics of granular materials
- Homogenization
- Variational principles
- Construction of compatible and equilibrated fields
- Local adjustment fields
- Comparison with Discrete Element Simulations
- Conclusions
Micromechanics of granular materials

Objective: relation macroscopic and microscopic characteristics

<table>
<thead>
<tr>
<th>Macroscopic level</th>
<th>Microscopic level</th>
</tr>
</thead>
<tbody>
<tr>
<td>• stress</td>
<td>• contact force</td>
</tr>
<tr>
<td>• strain</td>
<td>• contact relative displacement</td>
</tr>
</tbody>
</table>

Contact quantities

\[ \Delta^{pq} = U^q - U^p \]
Equilibrium conditions

\[ \sum_{q} f^{pq} = 0 \]

Compatibility conditions

\[ \sum_{S} \Delta^{RS} = 0 \]

Network connecting particle centres
Equilibrium conditions

\[ \sum_{q} f^{pq} = 0 \]

Compatibility conditions

\[ \sum_{S} \Delta^{RS} = 0 \]
**Micromechanical stress and strain**

\[
\sigma = \frac{1}{S} \sum_{c \in S} f^c l^c \\
\varepsilon = \frac{1}{S} \sum_{c \in S} \Delta^c h^c \\
I = \frac{1}{S} \sum_{c \in S} l^c h^c
\]

**Uniform fields**

**Kinematic form**  \( \Delta^c = \varepsilon \cdot l^c \)  (uniform strain)

**Static form**  \( f^c = \sigma \cdot h^c \)  (uniform stress)
Homogenization

Macroscopic level

Constitutive relation

Stress

Force

Homogenization

Strain

Relative displacement

Constitutive relation

Microscopic level

(contact)

Examples of homogenisation

\( \Delta^c = \epsilon \cdot I^c \)

\( f^c = \sigma \cdot h^c \)

\( \Delta^c = \zeta \epsilon \cdot I^c \)

\( f^c = \xi \sigma \cdot h^c \)
Contact constitutive relation

- No particle rotation
- Elastic constitutive relation
- No friction
- Bonded contacts

\[
\begin{align*}
  f_n^c &= k_n \Delta_n^c \\
  f_t^c &= k_t \Delta_t^c
\end{align*}
\]

\[
\begin{align*}
  f^c &= S^c \cdot \Delta^c \\
  \Delta^c &= C^c \cdot f^c
\end{align*}
\]
Energy principles

Minimum potential energy principle

\{ \Delta^* \}, \{ f^* \} \quad \text{compatible, not equilibrated}

\{ \Delta \}, \{ f \} \quad \text{compatible, equilibrated}

\[ U \leq U^* \]

With prescribed strain \( \Rightarrow \) upper bound to moduli

Minimum complementary energy principle

\{ f^* \}, \{ \Delta^* \} \quad \text{equilibrated, not compatible}

\{ f \}, \{ \Delta \} \quad \text{equilibrated, compatible}

\[ \hat{U} \leq \hat{U}^* \]

With prescribed stress \( \Rightarrow \) lower bound to moduli
**Compatible and equilibrated fields**

Compatible form  \[ \Delta^{pq} = U^q - U^p \]  (particle based)

Equilibrated form  \[ f^{RS} = \Phi^S - \Phi^R \]  (polygon based)

\[ \sum_{q} f^{pq} = f^{pa} + f^{pb} + f^{pc} + f^{pd} \]

\[ = [ \Phi^D - \Phi^A ] + [ \Phi^A - \Phi^B ] + [ \Phi^B - \Phi^C ] + [ \Phi^C - \Phi^D ] = 0 \]
Local adjustment fields

Assume: overall behaviour = uniform field + local adjustment

Kinematic formulation

\[ U^p = U^m(X^p; \varepsilon) + u^p \]
\[ U^q = U^m(X^q; \varepsilon) \]
\[ \Delta^{pq} = \varepsilon \cdot l^{pq} - u^p \]

Require (approximate) equilibrium

\[ \sum_q f^{pq} = 0 \]
\[ \left\{ \sum_q S^{pq} \right\} \cdot u^p = \sum_q \left\{ S^{pq} \cdot \varepsilon \cdot l^{pq} \right\} \]

Static formulation

\[ \Phi^R = \Phi^m(X^R; \sigma) + \phi^R \]
\[ \Phi^S = \Phi^m(X^S; \sigma) \]

Require (approximate) compatibility

\[ \sum_S \Delta^{RS} = 0 \]
\[ \left\{ \sum_S C^{RS} \right\} \cdot \phi^R = \sum_S \left\{ C^{RS} \cdot \sigma \cdot h^{RS} \right\} \]
Results from Discrete Element simulations

**Loose system**

- **Uniform strain**
- **Uniform stress**
- **LAF bounds**

**Dense system**

- **Uniform stress**

**Relative displacements**

- **Simulation, tangential**
- **Bound, tangential**
- **Bound, normal**
- **Simulation, normal**

**Forces**

- **Simulation, tangential**
- **Bound, tangential**

\[ \Delta_n^c = \zeta_n (\varepsilon \cdot I^c) \cdot n \]

\[ \Delta_t^c = \zeta_t (\varepsilon \cdot I^c) \cdot t \]

\[ f_n^c = \xi_n (\sigma \cdot h^c) \cdot n \]

\[ f_t^c = \xi_t (\sigma \cdot h^c) \cdot t \]
Conclusions

- construction of equilibrated contact force fields
- uniform field homogenization leads to upper and lower bounds for moduli
- uniform strain assumption is only correct for dense systems
- homogenization based on mean field and fluctuation that is determined by local environment
  - tight bounds for moduli
  - good predictions of relative displacement and force fields at contacts