Multi-Level Wave-Ray Method for 2D Helmholtz Equation

P.C. Verburg June 23 2010
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Introduction
Helmholtz equation

β Helmholtz equation describes waves.
β Applications in acoustics, mechanics, thermodynamics and electrical engineering.
β This research: acoustics.
Noise pollution is an important problem in daily life.
Tool is needed to evaluate sound-mitigation measures.
Generally no analytical solution is available.
Therefore numerical methods are required.
Introduction
Sound waves

Sound waves cause small-amplitude perturbations:
\[ p = p_0 + p' \quad \rho = \rho_0 + \rho' \quad u_0 = 0 \]
\[ p' \ll p_0 \quad \rho \ll \rho_0 \]

No external force fields/heat sources.
No viscosity and heat conduction.
Isentropic flow.
Calorically perfect gas.

Governing equations for mass, momentum and energy conservation are applied.
Introduction
Separation of variables

Standing wave

Wave equation:
\[ \frac{\partial^2 u}{\partial t^2} = c^2 \nabla^2 u \]

Separation of variables:
\[ u(x, t) = \hat{u}(x) e^{i\omega t} \]

Helmholtz equation describes the space-dependence of sound waves.
Introduction
Helmholtz equation

- **Helmholtz equation (elliptic PDE):**

\[ \nabla^2 u + k^2 u = f \]

- Describes space-dependence of pressure/density/velocity perturbations of standing waves.

- Wavenumber \( k \): frequency of the waves in the solution (this research: constant \( k \)).

\[ k = \frac{2\pi}{\lambda} \]

- Difficult to solve numerically for many wave-periods in domain.
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- **Finite difference method**
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Finite difference method

Introduction

\[ \nabla^2 u + k^2 u \]

Discrete operator:

\[ L^h \langle u \rangle_{i,j} = \frac{u_{i-1,j}^h - 2u_{i,j}^h + u_{i+1,j}^h}{h_x^2} + \frac{u_{i,j-1}^h - 2u_{i,j}^h + u_{i,j+1}^h}{h_y^2} + k^2 u_{i,j}^h \]

Second-order accurate.
Finite difference method
Single grid iterative solver

Residual: \[ r_{i,j}^h = f_{i,j}^h - L^h \langle \tilde{u}^h \rangle_{i,j} \]

Difference between right-hand side of equation and discrete operator applied to approximate solution.

Converged solution: \[ r_{i,j}^h = 0 \quad \forall i, j \in \Omega \]

Iterative solver: adjust solution point by point such that the residual is reduced or eliminated.

Sweep across all points: relaxation.
Finite difference method
Single grid iterative solver - 1D demonstration

Laplace equation:
\[
\frac{d^2 u}{dx^2} = 0
\]

Computation time for 2D situations:
\[
O(N^2)
\]
Finite difference method
Conclusion

- Single grid iterative solvers are unable to remove components with a low frequency/large wavelength relative to the grid.
- Such components are insensitive to local relaxations because only a small part of the wave is taken into account by the discrete operator.
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Multi-level method

Introduction

- Multi-level principle: represent every frequency component in the error on a scale at which it can be accurately described and efficiently solved.

- Multi-level solvers have been successfully applied to several elliptic PDE’s such as the Laplace equation.

- Ideally, computation time: $O(N)$
Multi-level method
V-cycle

- Highest level should represent solution with the desired accuracy.
- Lower levels are a means to accelerate the convergence speed.
- Components with a low frequency relative to a fine grid have a high frequency relative to a coarse grid.

<table>
<thead>
<tr>
<th>Mesh size</th>
<th>Level</th>
</tr>
</thead>
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<tr>
<td>h=H</td>
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</tr>
<tr>
<td>h=2H</td>
<td>2</td>
</tr>
<tr>
<td>h=4H</td>
<td>1</td>
</tr>
</tbody>
</table>
Multi-level method
Multi-level solver - 1D demonstration

Laplace equation:
\[
\frac{d^2 u}{dx^2} = 0
\]

Computation time for 2D situations:
\[O(N)\]
A multi-level solver should be able to efficiently remove all frequency components in the error to obtain required performance.

However, standard multi-level methods fail to remove errors with a frequency close to the solution for the Helmholtz equation.

Characteristic components

\[ k_x^2 + k_y^2 \approx k^2 \]
Multi-level method
Removing characteristic components on fine grids

Ratio between error before/after relaxation:

\[ k_x^2 + k_y^2 \approx k^2 \]

- Locally the solution satisfies the discrete operator.
- Low frequency relative to the grid.
Multi-level method
Removing characteristic components on coarse grids

A phase error always occurs when discretising a continuous problem.
Phase error accumulates with every wave-period in the domain.
Phase error becomes larger on coarse grids.
Characteristic components in the error with a frequency close to the solution cannot be removed on both fine and coarse grids.

$$k_x^2 + k_y^2 \approx k^2$$

To make the multi-level approach succeed for the Helmholtz equation, a separate treatment is necessary for the characteristic components.

Multi-level cycle for Helmholtz equation: wave-cycle.
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Wave-Ray algorithm

Introduction

- Wave-cycle shows good performance except for characteristic frequencies.

- Performance of wave-cycle should be retained for other frequencies.

- Return to principle behind multi-level method: describe all components in the error in a way that allows efficient removal.

- Brandt and Livshits: characteristic components are described separately for each wave-propagation direction (wave-ray method).
Wave-Ray algorithm
Introduction

- Frequency of characteristic components is known.
- Wave propagation direction (ray) is fixed.
- Therefore only the amplitude has to be described.
- Amplitude changes very gradually, coarse grid is sufficient, no phase errors.
Wave-Ray algorithm

Ray-directions

- For a numerical method the number of rays should be limited.
- Wave-ray method uses 8 rays.
- 90% correction.
Wave-Ray algorithm
Ray equations

- Only an approximate solution has been obtained using a standard multi-level cycle:

\[ \nabla^2 \tilde{u} + k^2 \tilde{u} = f - r \quad \text{with} \quad r \text{ the residual} \]

- Because the Helmholtz equation is a linear PDE, a correction can be computed:

\[ u = \tilde{u} + v \]

- By solving the equation:

\[ \nabla^2 v + k^2 v = r \]
Wave-Ray algorithm
Ray equations

However, \( \nabla^2 \nu + k^2 \nu = r \) is just as difficult to solve.
Therefore a new description of \( \nu \) and \( r \) is needed.
Represent characteristic components separately for each wave propagation direction:

\[
\nu = \sum_{i=1}^{8} a_i(\xi, \eta) e^{ik\xi} \quad r = \sum_{i=1}^{8} r_i(\xi, \eta) e^{ik\xi}
\]

\( \xi \): wave propagation direction

\[
\frac{\partial^2 a_i(\xi, \eta)}{\partial \xi^2} + \frac{\partial^2 a_i(\xi, \eta)}{\partial \eta^2} + 2i \frac{\partial}{\partial \xi} a_i(\xi, \eta) = r_i(\xi, \eta)
\]

Characteristic components are represented by smooth envelope functions, therefore no phase error.
Wave-Ray algorithm
Outline of the solver

1. Run standard multi-level solver (wave-cycle)
2. Split residual into different wave propagation directions
3. Correct wave-cycle solution
4. Solve ray equations
Wave-Ray algorithm
Separation and merging

β Separation procedure for the wave-cycle residual:

\[ r(x, y) \rightarrow \sum_{i=1}^{8} r_i(\xi, \eta) e^{ik\xi} \]

β Solve ray equations:

\[
\frac{\partial^2}{\partial \xi^2} a_i(\xi, \eta) + \frac{\partial^2}{\partial \eta^2} a_i(\xi, \eta) + 2i \frac{\partial}{\partial \xi} a_i(\xi, \eta) = r_i(\xi, \eta)
\]

β Merge corrections:

\[ u(x, y) = \tilde{u}(x, y) + \sum_{i=1}^{8} a_i(\xi, \eta) e^{ik\xi} \]
Wave-Ray algorithm
Separation and merging

Residual → Rotate → Separation/solve ray equation

Residual → Rotate → Add correction
Wave-Ray algorithm
Boundary conditions

- Ray equations are solved on a larger domain than the original problem.
- Allows Sommerfeld condition to be introduced: waves that propagate through the domain without obstructions.
- Near-field solution can be obtained without computing far-field solution.
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Results
Sound-sources outside the domain

Solution of problem with 16 plane wave sources outside domain.

Grid: 512x512
Results

Convergence speed

- Convergence speed remains constant up to machine accuracy.
- About 25 wave-ray cycles required for full convergence.
Results

Convergence speed

- Convergence speed independent of number of grid points.
- $O(N)$ computation time to solve discrete Helmholtz problem.
Results

Convergence speed

- Convergence speed independent of number of wave-periods in domain.
- $kd$: product of wavenumber and domain size.
Results
Sound-sources inside the domain

- Solution of problem with a line source inside the domain.
- Identical performance compared with sources outside the domain.

\begin{align*}
\text{Line source}
\end{align*}
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Conclusions and recommendations

Conclusions

- A very efficient multi-level solver has been developed for the 2D Helmholtz equation.

- Several types of boundary conditions have been introduced.

- It is expected that the wave-ray method can be developed into a solver suitable for applications in engineering.
Conclusions and recommendations

Recommendations

- Investigate how the wave-ray solver can deal with additional boundary conditions.

- Extend the solver from 2D to 3D.

- Adapt method to problems with strongly varying wavenumbers.

- Further investigate accuracy of the wave-ray method.
Thank you for your attention