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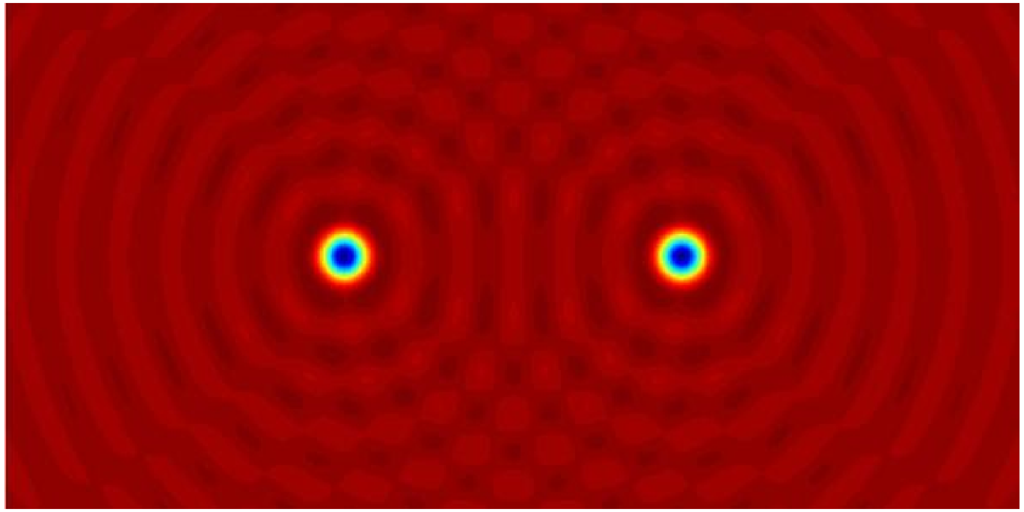
Multi-Level Wave-Ray Method for 2D Helmholtz Equation

P.C. Verburg June 23 2010

Contents

Introduction
Helmholtz equation

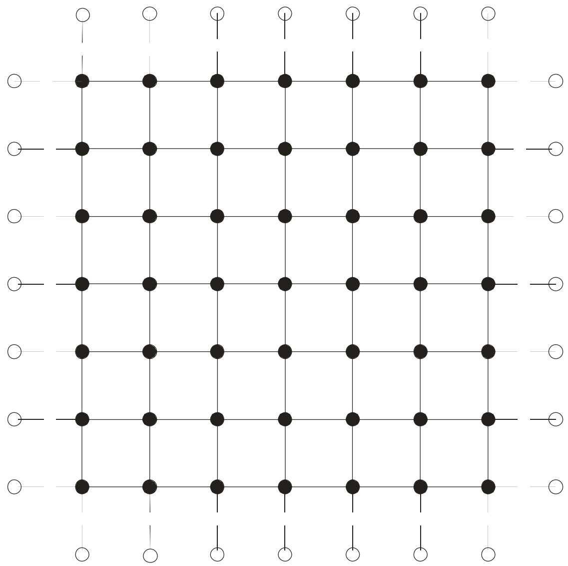
- Helmholtz equation describes waves.
- Applications in acoustics, mechanics, thermodynamics and electrical engineering.
- This research: acoustics.



Introduction
Numerical methods

- Noise pollution is an important problem in daily life.
- Tool is needed to evaluate sound-mitigation measures.
- Generally no analytical solution is available.
- Therefore numerical methods are required.

Source: Rijkswaterstaat



Introduction
Sound waves

- Sound waves cause small-amplitude perturbations:

$$p = p_0 + p' \qquad \rho = \rho_0 + \rho' \qquad \vec{u}_0 = \vec{0}$$

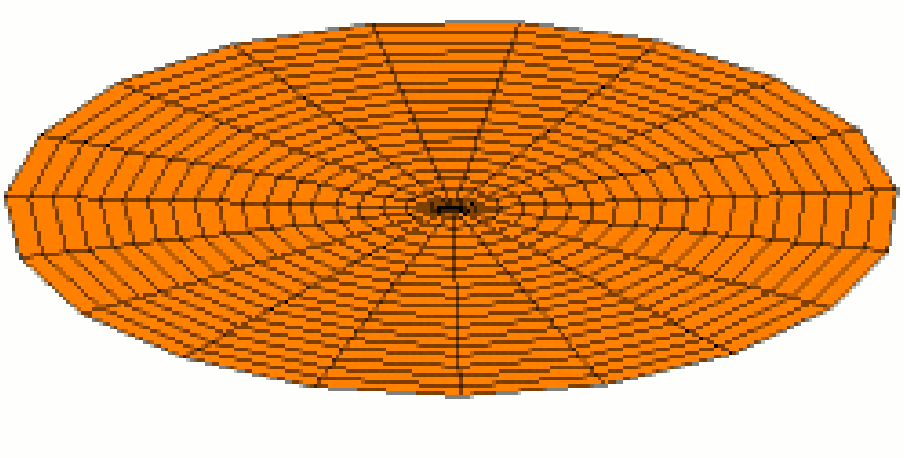
$$p' \ll p_0 \qquad \rho \ll \rho_0$$

- No external force fields/heat sources.
- No viscosity and heat conduction.
- Isentropic flow.
- Calorically perfect gas.

- Governing equations for mass, momentum and energy conservation are applied.

Introduction
Separation of variables

Standing wave



Wave equation:

$$\frac{\partial^2 u}{\partial t^2} = c^2 \nabla^2 u$$

Separation of variables:

$$u(\vec{x}, t) = \hat{u}(\vec{x})\, e^{i\omega t}$$

Space Time

- Helmholtz equation describes the space-dependence of sound waves.

Introduction
Helmholtz equation

- **Helmholtz equation (elliptic PDE):**

$$\nabla^2 u + k^2 u = f$$

- Describes space-dependence of pressure/density/velocity perturbations of standing waves.

- Wavenumber ***k*****:** frequency of the waves in the solution (this research: constant *k*).

$$k = \frac{2p}{l}$$

- Difficult to solve numerically for many wave-periods in domain.

Contents

Finite difference method
Introduction

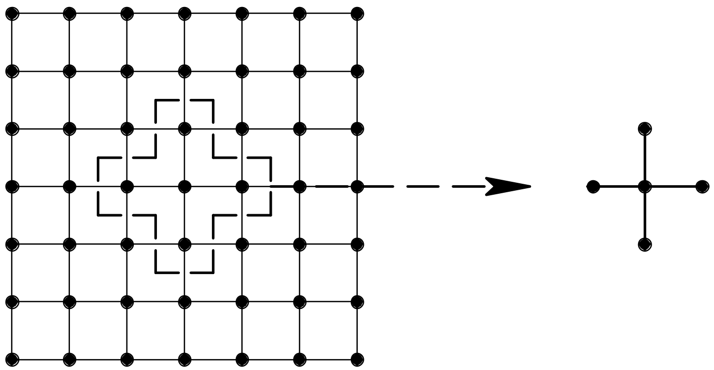
- Left-hand side of Helmholtz equation:

$$\nabla^2 u + k^2 u$$

- Discrete operator:

$$L^h \langle u \rangle_{i,j} = \frac{u^h_{i-1,j} - 2u^h_{i,j} + u^h_{i+1,j}}{h_x^2} + \frac{u^{\mathrm{h}}_{\mathrm{i,\,j\text{-}1}} - 2u^h_{i,j} + u^h_{i,j+1}}{h_y^2} + k^2 u^h_{i,j}$$

- Second-order accurate.



Finite difference method

Single grid iterative solver

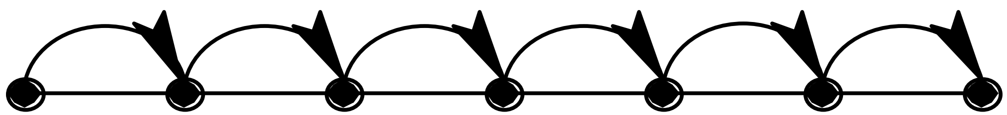
- Residual: $r_{i,j}^{h} = f_{i,j}^{h} - L^{h}\left\langle \tilde{u}^{h} \right\rangle_{i,j}$

- Difference between right-hand side of equation and discrete operator applied to approximate solution.

- Converged solution: $r_{i,j}^{h} = 0 \quad \forall i, j \in \Omega$

- Iterative solver: adjust solution point by point such that the residual is reduced or eliminated.

- Sweep across all points: **relaxation**.

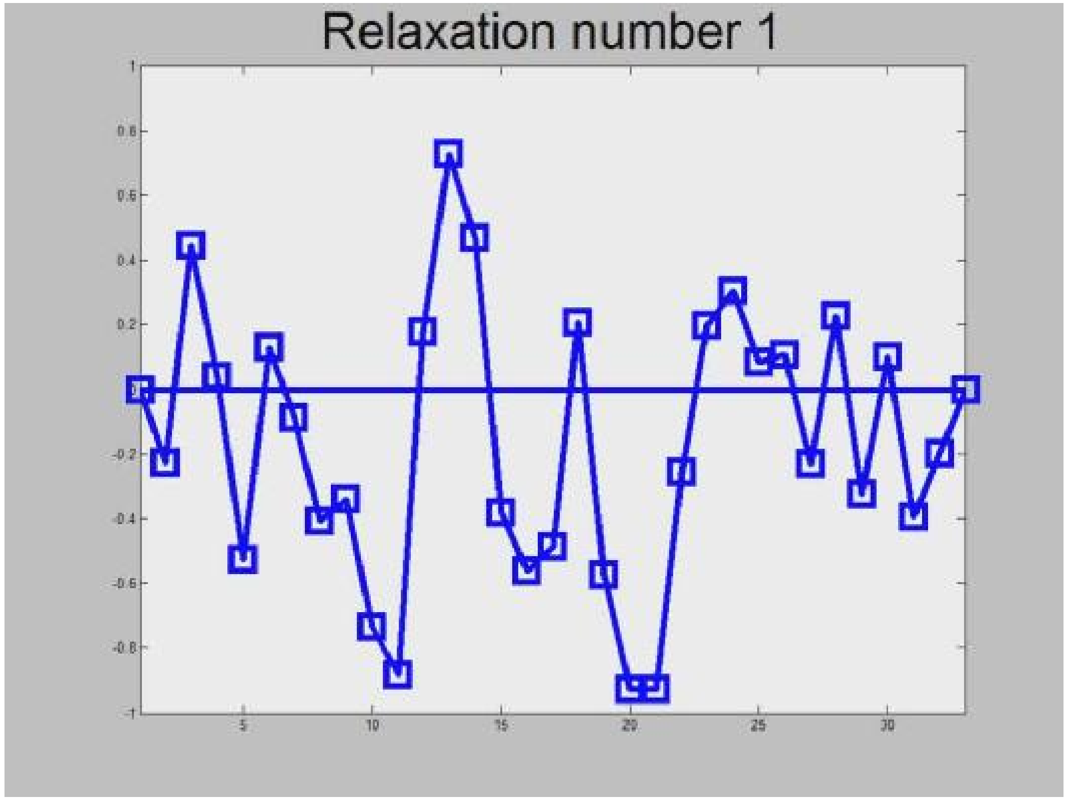


Finite difference method

Single grid iterative solver - 1D demonstration

Laplace equation:

$$\frac{d^2u}{dx^2} = 0$$

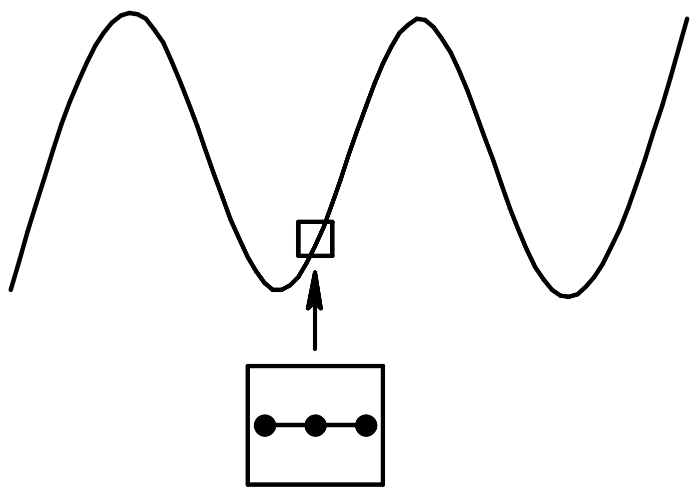


Computation time for 2D situations:

$$O\left(N^2\right)$$

Finite difference method
Conclusion

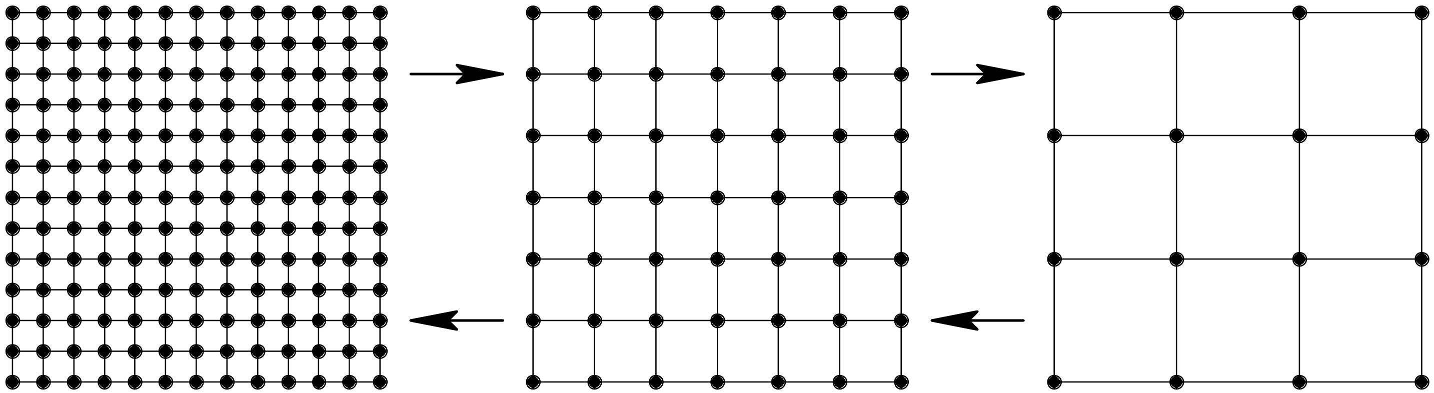
- Single grid iterative solvers are unable to remove components with a **low frequency/large wavelength relative to the grid**.
- Such components are insensitive to local relaxations because only a small part of the wave is taken into account by the discrete operator.



Contents

Multi-level method
Introduction

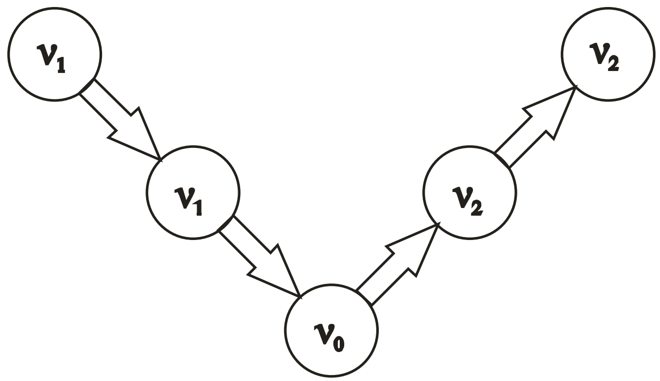
- Multi-level principle: represent every frequency component in the error on a scale at which it can be accurately described and efficiently solved.
- Multi-level solvers have been successfully applied to several elliptic PDE's such as the Laplace equation.
- Ideally, computation time: $O(N)$



Multi-level method
V-cycle

- Highest level should represent solution with the desired accuracy.
- Lower levels are a means to accelerate the convergence speed.
- Components with a low frequency relative to a fine grid have a high frequency relative to a coarse grid.

Mesh size	Level
h=H	3
h=2H	2
h=4H	1

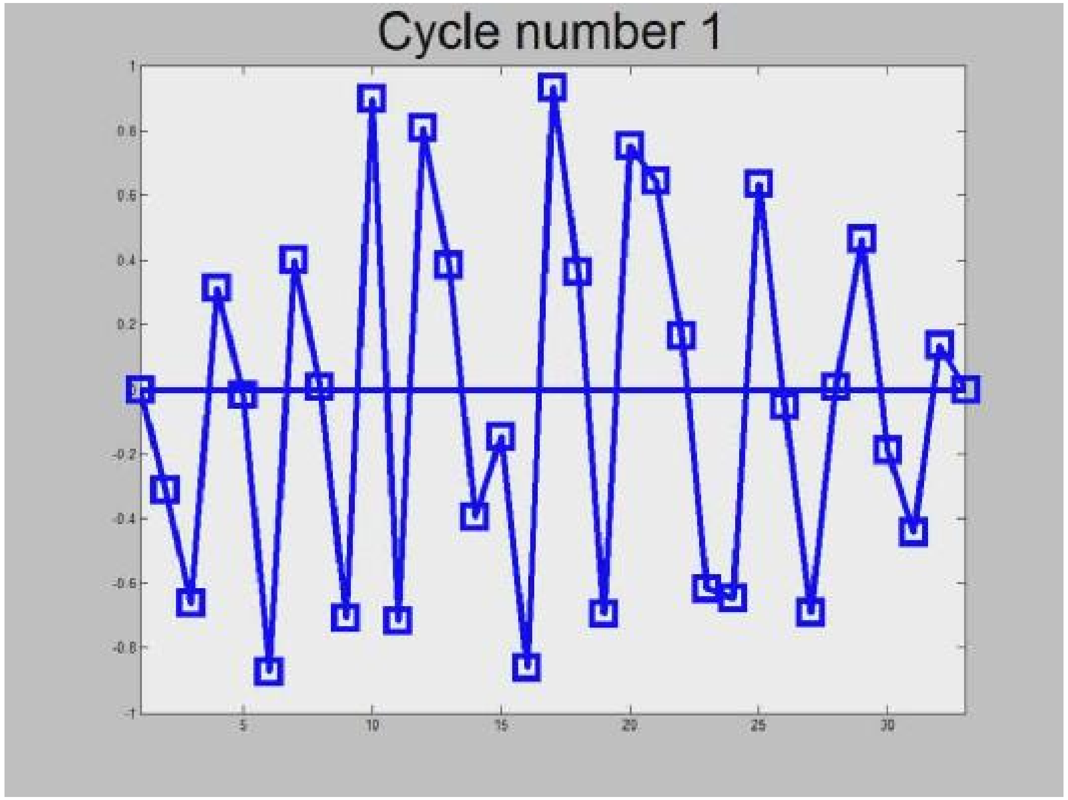


Multi-level method

Multi-level solver - 1D demonstration

Laplace equation:

$$\frac{d^2u}{dx^2} = 0$$



Computation time for 2D situations:

$O(N)$

Multi-level method
Application to the Helmholtz equation

- A multi-level solver should be able to efficiently remove all frequency components in the error to obtain required performance.

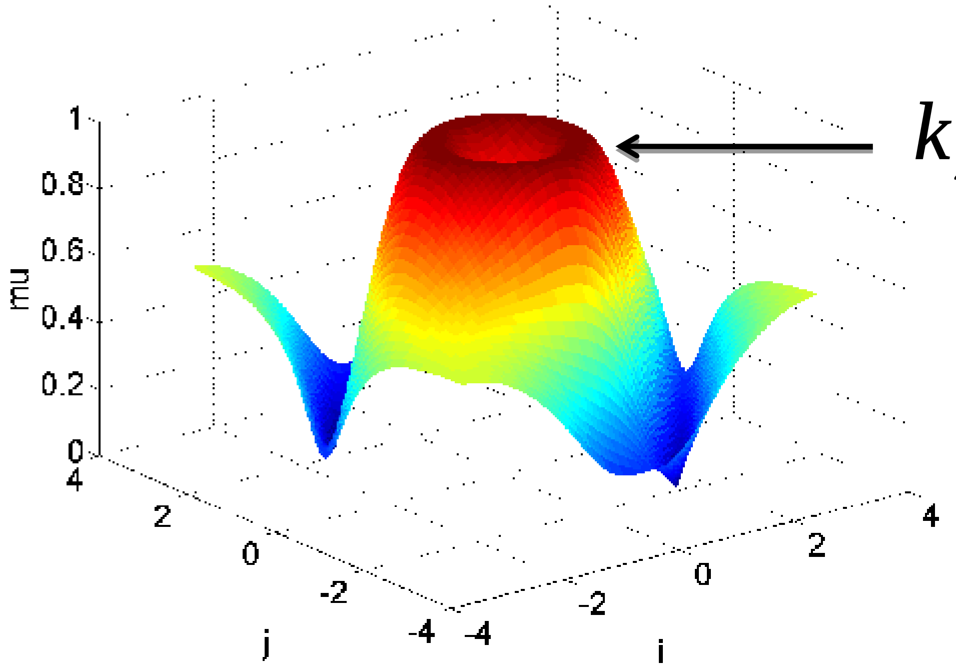
- However, standard multi-level methods fail to remove errors with a frequency close to the solution for the Helmholtz equation.

- **Characteristic components** $k_x^2 + k_y^2 \approx k^2$

Multi-level method

Removing characteristic components on fine grids

Ratio between error before/after relaxation:



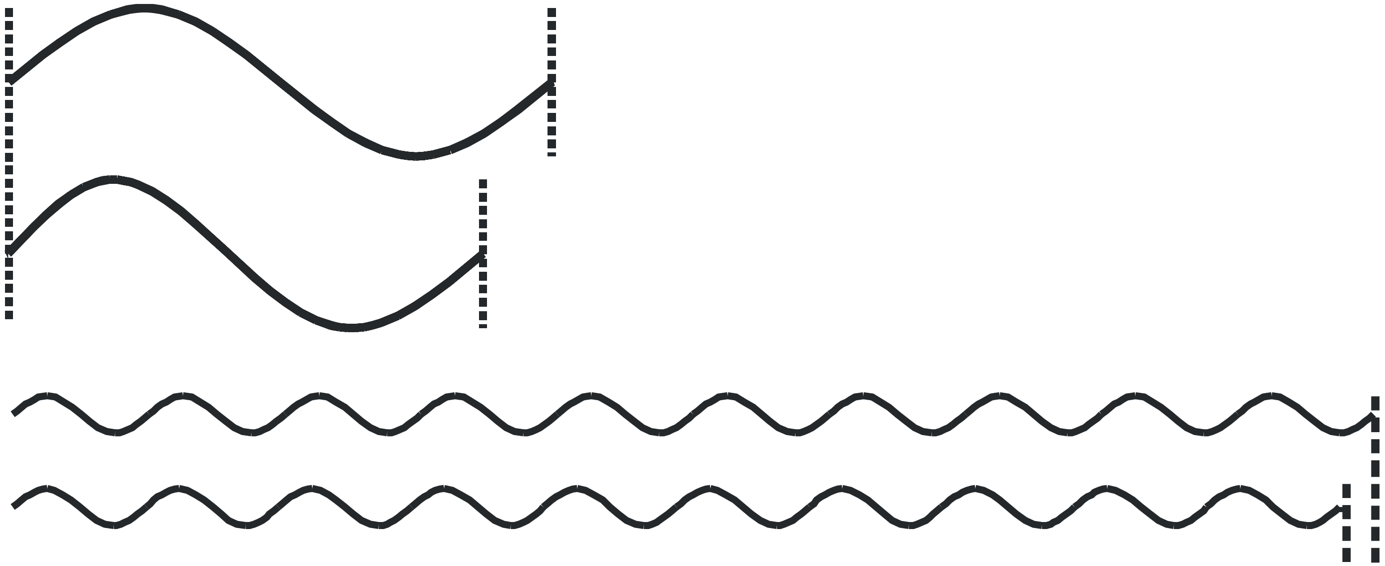
$$k_x^2 + k_y^2 \approx k^2$$

- Locally the solution satisfies the discrete operator.
- Low frequency relative to the grid.

Multi-level method

Removing characteristic components on coarse grids

- A phase error always occurs when discretising a continuous problem.
- Phase error accumulates with every wave-period in the domain.
- Phase error becomes larger on coarse grids.



Multi-level method

Conclusion

- Characteristic components in the error with a frequency close to the solution cannot be removed on both fine and coarse grids.

$$k_x^2 + k_y^2 \approx k^2$$

- To make the multi-level approach succeed for the Helmholtz equation, a separate treatment is necessary for the characteristic components.

- Multi-level cycle for Helmholtz equation: **wave-cycle.**

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- Introduction
- Finite difference method
- Multi-level method
- **Wave-Ray algorithm**
- Results
- Conclusions and recommendations

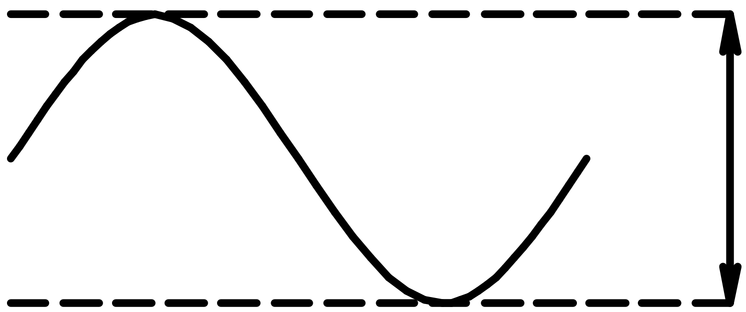
Wave-Ray algorithm

Introduction

- **Wave-cycle** shows good performance except for characteristic frequencies.
- Performance of wave-cycle should be retained for other frequencies.
- Return to principle behind multi-level method: describe all components in the error in a way that allows efficient removal.
- Brandt and Livshits: characteristic components are described separately for each wave-propagation direction (**wave-ray** method).

Wave-Ray algorithm
Introduction

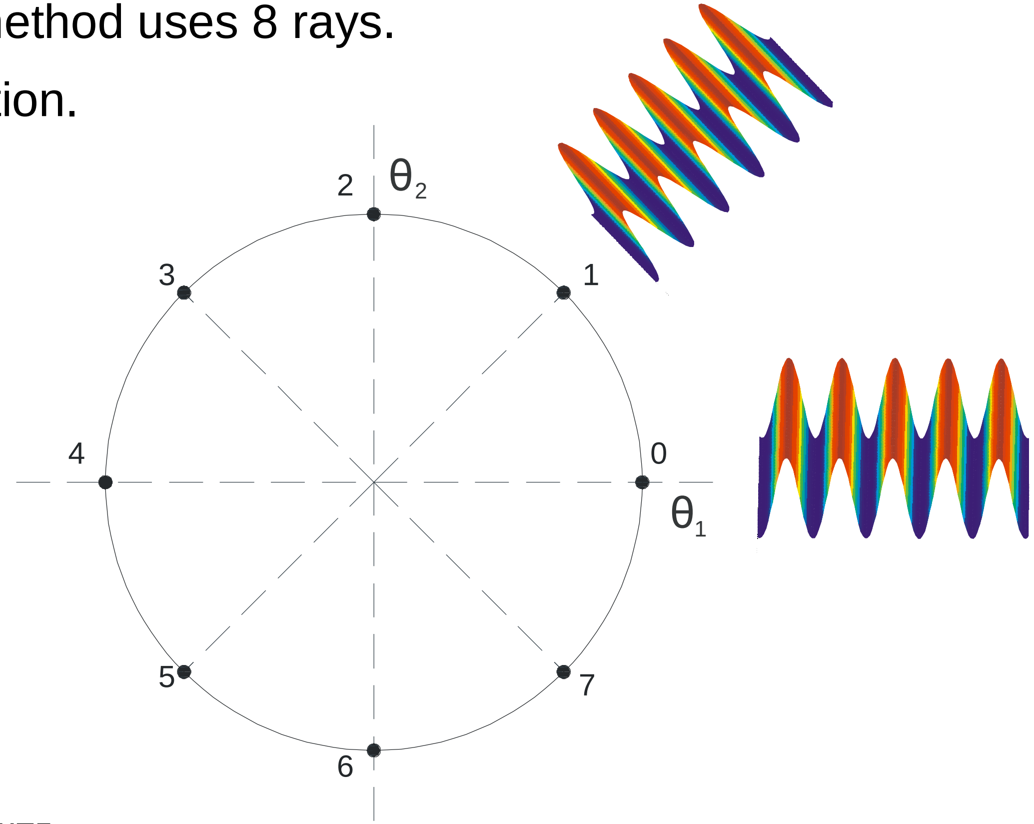
- Frequency of characteristic components is known.
- Wave propagation direction (ray) is fixed.
- Therefore only the amplitude has to be described.
- Amplitude changes very gradually, coarse grid is sufficient, no phase errors.



Wave-Ray algorithm

Ray-directions

- For a numerical method the number of rays should be limited.
- Wave-ray method uses 8 rays.
- 90% correction.



Wave-Ray algorithm
Ray equations

- Only an approximate solution has been obtained using a standard multi-level cycle:

$$\nabla^2 \tilde{u} + k^2 \tilde{u} = f - r$$ with r the residual

- Because the Helmholtz equation is a linear PDE, a correction can be computed:

$$u = \tilde{u} + v$$

- By solving the equation:

$$\nabla^2 v + k^2 v = r$$

Wave-Ray algorithm

Ray equations

- However, $\nabla^2 v + k^2 v = r$ is just as difficult to solve.
- Therefore a new description of *v* and *r* is needed.
- Represent characteristic components separately for each wave propagation direction:

$$v = \sum_{i=1}^{8} \boxed{a_i(x,h)} e^{ikx} \qquad r = \sum_{i=1}^{8} \boxed{r_i(x,h)} e^{ikx}$$

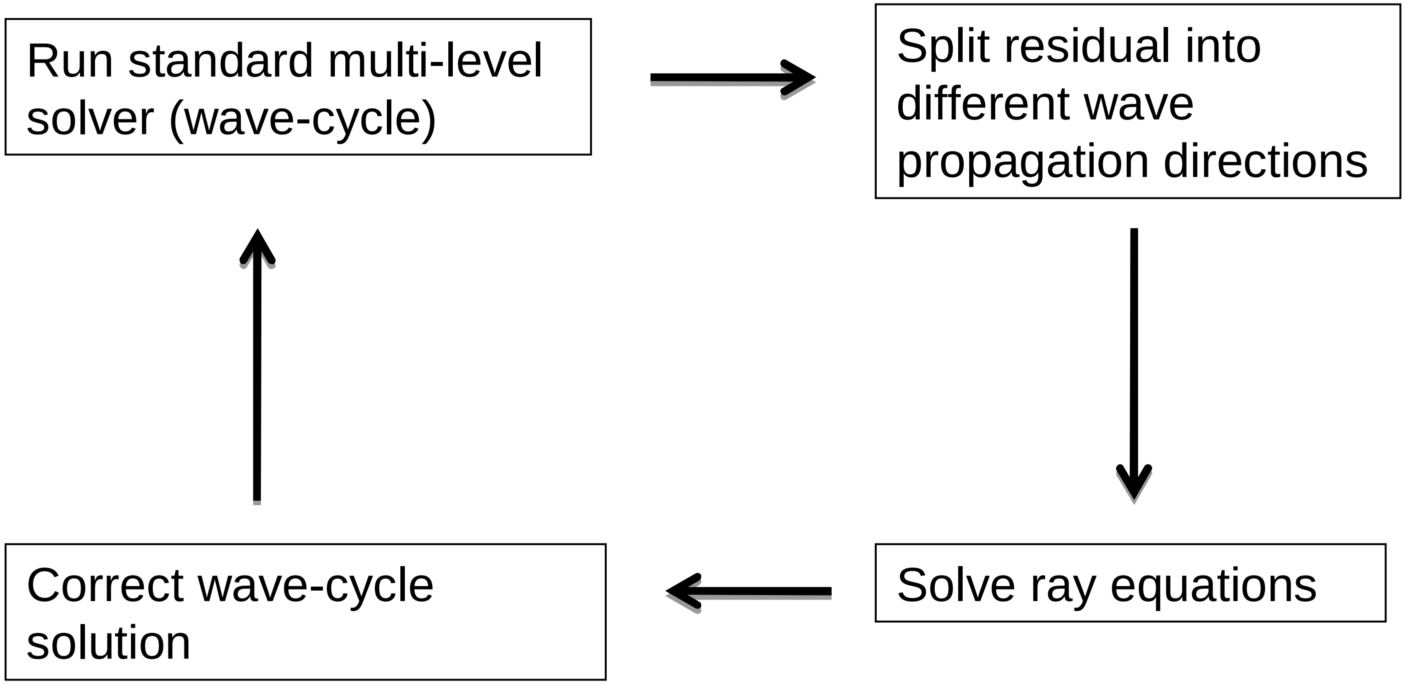
- ξ: wave propagation direction

$$\frac{\partial^2}{\partial x^2} \boxed{a_i(x,h)} + \frac{\partial^2}{\partial h^2} \boxed{a_i(x,h)} + 2i \frac{\partial}{\partial x} \boxed{a_i(x,h)} = \boxed{r_i(x,h)}$$

- Characteristic components are represented by smooth envelope functions, therefore no phase error.

Wave-Ray algorithm

Outline of the solver



Wave-Ray algorithm
Separation and merging

- Separation procedure for the wave-cycle residual:

$$r(x,y) \longrightarrow \sum_{i=1}^{8} \boxed{r_i(\xi,\eta)}\, e^{\imath k\xi}$$

- Solve ray equations:

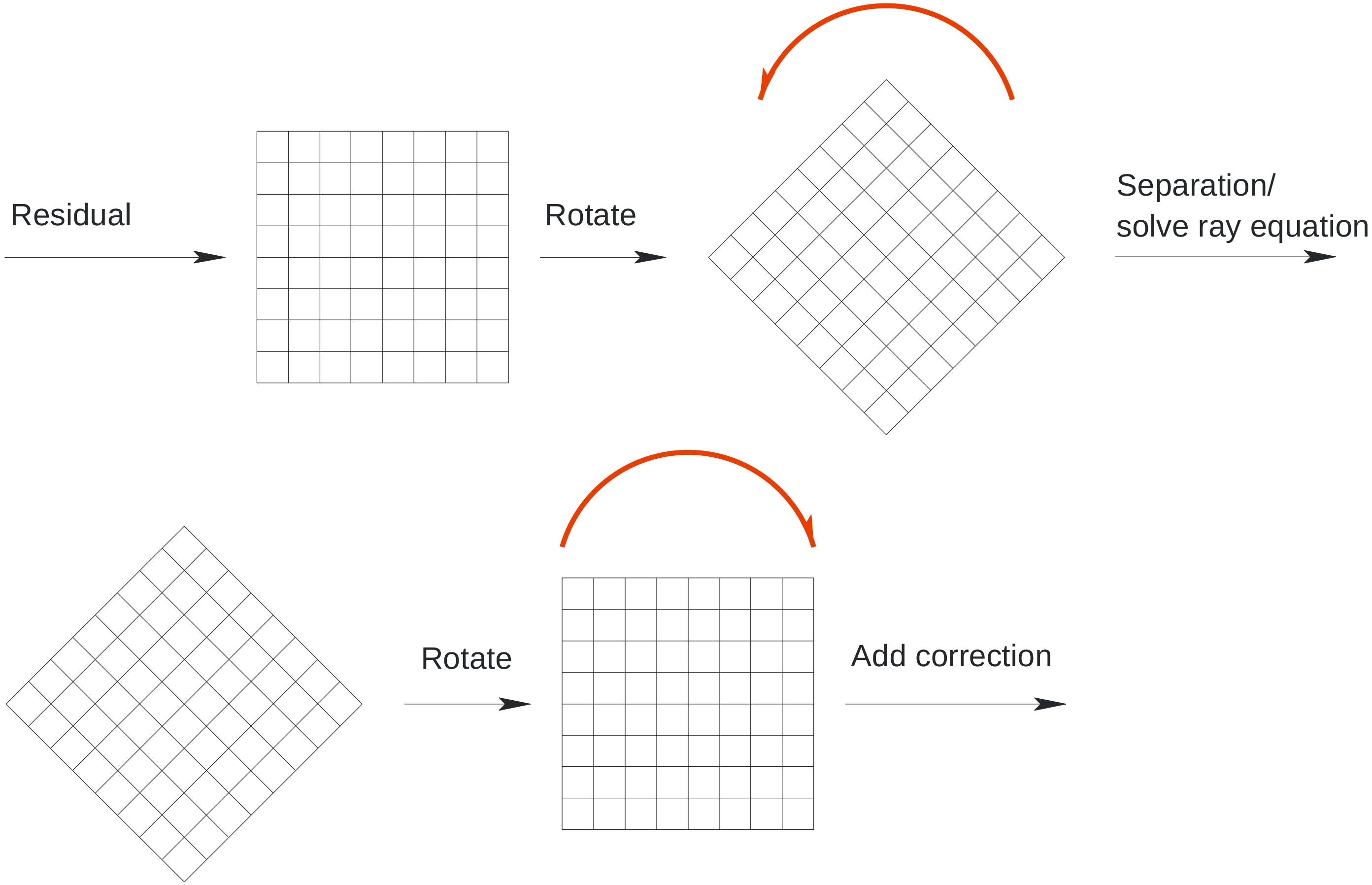
$$\frac{\partial^2}{\partial x^2}\boxed{a_i(x,h)} + \frac{\partial^2}{\partial h^2}\boxed{a_i(x,h)} + 2i\frac{\partial}{\partial x}\boxed{a_i(x,h)} = \boxed{r_i(x,h)}$$

- Merge corrections:

$$u(x,y) = \tilde{u}(x,y) + \sum_{i=1}^{8} \boxed{a_i(x,h)}\, e^{ikx}$$

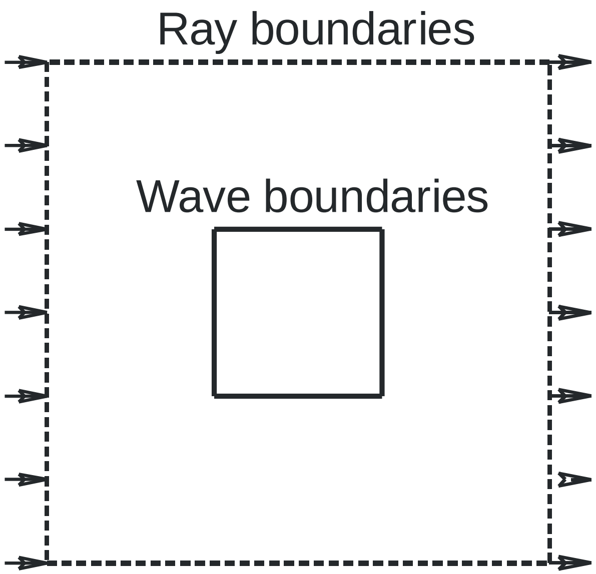
Wave-Ray algorithm

Separation and merging



Wave-Ray algorithm
Boundary conditions

- Ray equations are solved on a larger domain than the original problem.
- Allows Sommerfeld condition to be introduced: waves that propagate through the domain without obstructions.
- Near-field solution can be obtained without computing far-field solution.

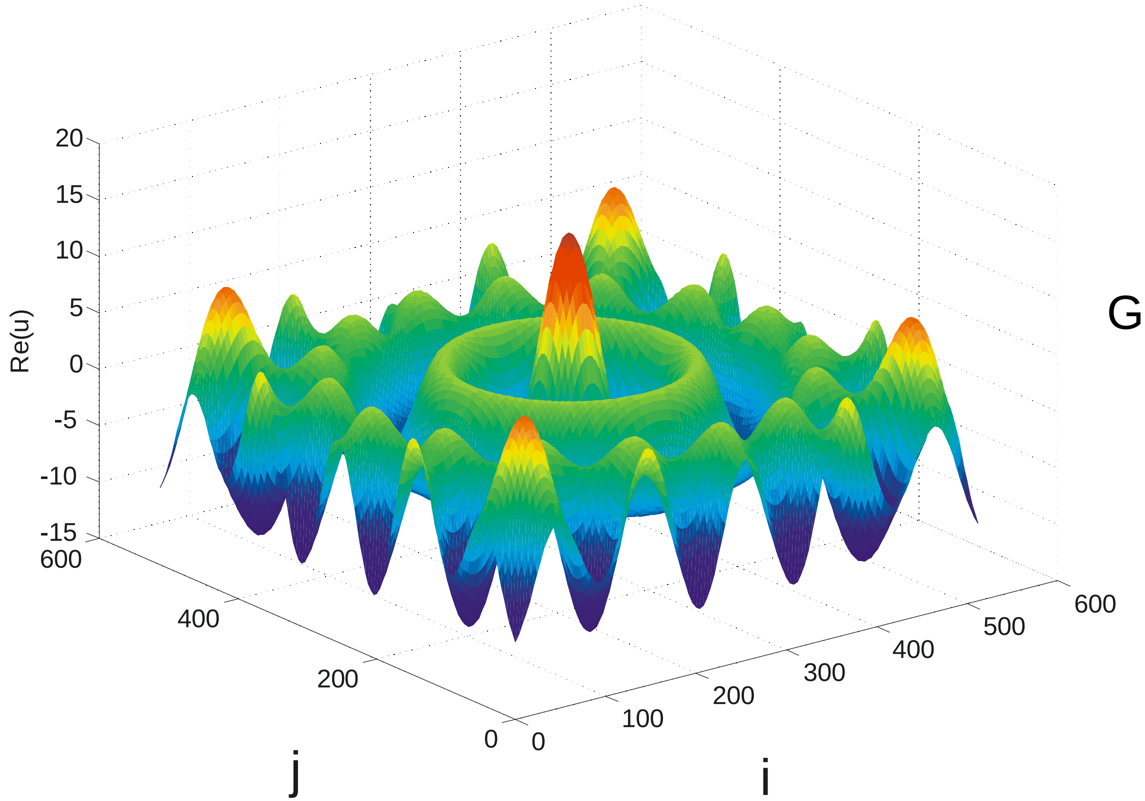


Contents

Results

Sound-sources outside the domain

- Solution of problem with 16 plane wave sources outside domain.

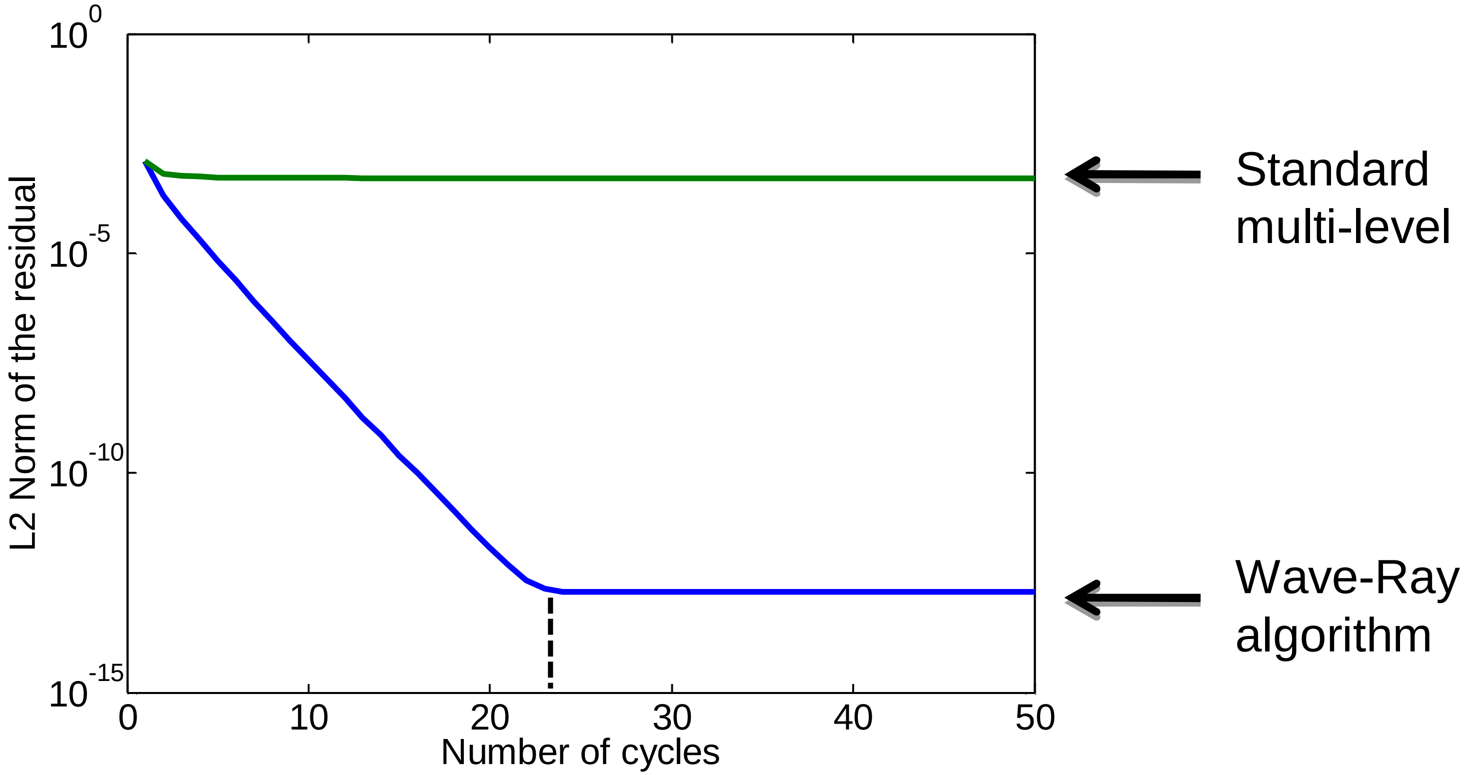


Grid: 512x512

Results

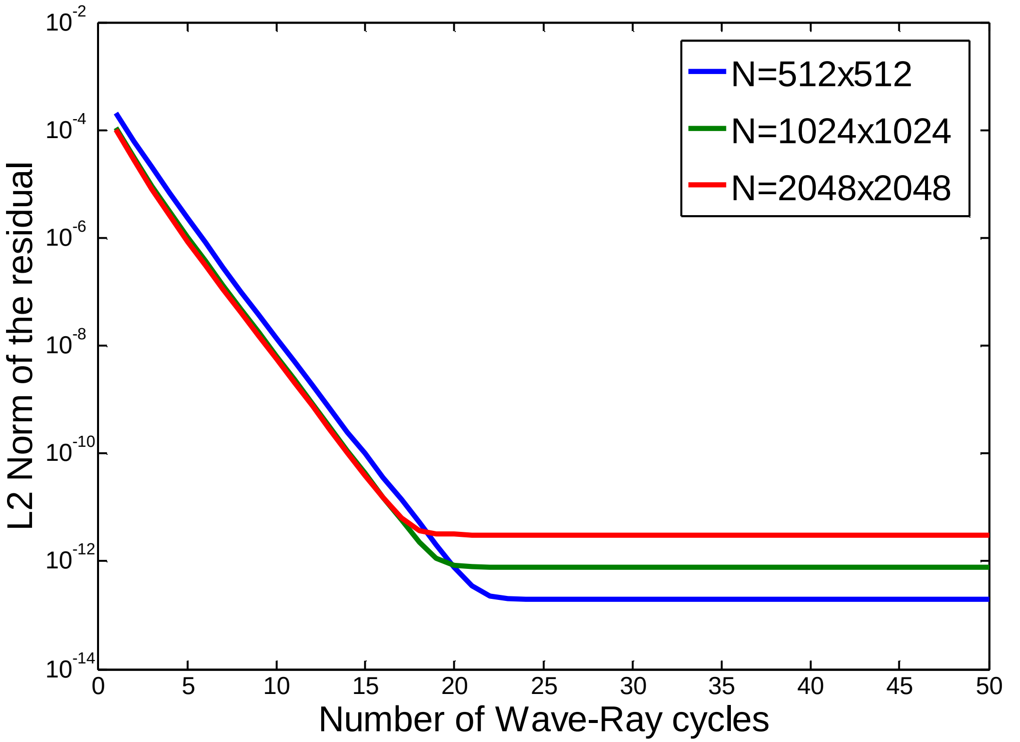
Convergence speed

- Convergence speed remains constant up to machine accuracy.
- About 25 wave-ray cycles required for full convergence.



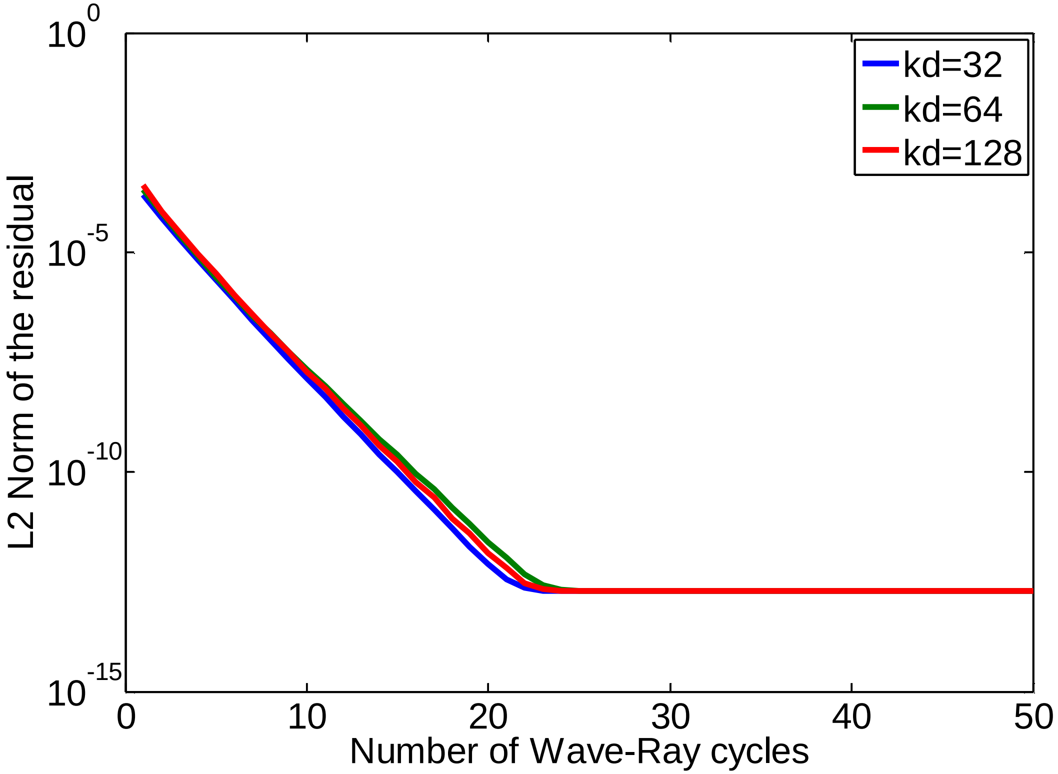
Results
Convergence speed

- Convergence speed independent of number of grid points.
- $O(N)$ computation time to solve discrete Helmholtz problem.



Results
Convergence speed

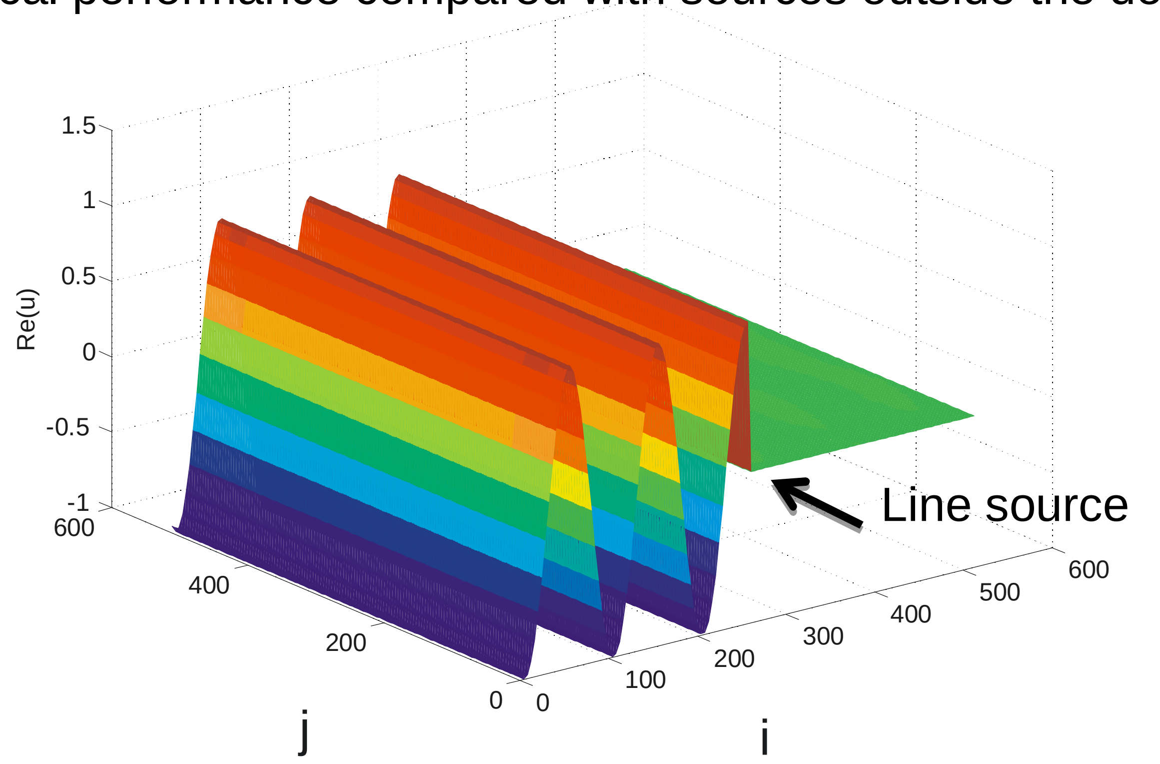
- Convergence speed independent of number of wave-periods in domain.
- kd: product of wavenumber and domain size.



Results

Sound-sources inside the domain

- Solution of problem with a line source inside the domain.
- Identical performance compared with sources outside the domain.



Contents

- Introduction
- Finite difference method
- Multi-level method
- Wave-Ray algorithm
- Results
- **Conclusions and recommendations**

Conclusions and recommendations
Conclusions

- A very efficient multi-level solver has been developed for the 2D Helmholtz equation.
- Several types of boundary conditions have been introduced.
- It is expected that the wave-ray method can be developed into a solver suitable for applications in engineering.

Conclusions and recommendations

Recommendations

- Investigate how the wave-ray solver can deal with additional boundary conditions.

- Extend the solver from 2D to 3D.

- Adapt method to problems with strongly varying wavenumbers.

- Further investigate accuracy of the wave-ray method.

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Thank you for your attention